S100 M

THE OPEN UNIVERSITY

Science Foundation Course

Mathematics for the Foundation Course in Science

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26	·4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	4314
27	·4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	4472
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29	·4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	4771
30	·4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	4914
31	·4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	5051
32	·5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	5185
33	·5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	5315
34	·5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	5441
35	·5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	5563
36	·5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	5682
37	·5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	5798
38	·5798	5809	5821	5382	5843	5855	5866	5877	5888	5899	5911
39	·5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	6021
40	·6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	6128
41	·6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	6232
42	·6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	6335
43	·6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	6435
44	·6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	6532
45	·6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	6628
46	·6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	6721
47	·6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	6812
48	·6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	6902
49	·6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	6990
50	·6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	7076
51	·7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	7160
52	·7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	7243
53	·7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	7324
54	·7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	7404

The Open University

Foundation Course in Science

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Contents

Glossary of symbols used in this book in the order in which they appear

$\sqrt[q]{a}$	positive qth root of a: $\sqrt[q]{a} = a^{1/q}$
$\log_a x$	the logarithm of the number x to base a
$\leqslant$	less than or equal to
$<$	less than
$\bar{n}$	"bar n" means: $-n$
$\text{antilog}_a\, x$	the antilogarithm of the number x to base a
$\approx$	is approximately equal to
$\ll$	is very much smaller than
$\lvert a\rvert$	the modulus of a: $\lvert a\rvert = a$, if a is positive $\lvert a\rvert = -a$, if a is negative
$\sum_{i=1}^{n} a_i =$	$a_1 + a_2 + a_3 + \cdots + a_n$
$\propto$	is directly proportional to
$a \cdot b$	$a \times b$
ψ	psi (Greek)
$\pm$	plus or minus
π	pi (Greek)
$\neq$	is not equal to
α	alpha (Greek)
β	beta (Greek)
γ	gamma (Greek)
θ	theta (Greek)
$\angle ABC$	the angle at B between AB and BC
$\angle C$	the angle at C
$\mathbf{a}$	vector of magnitude a
Δ	delta (Greek)
Δx	a difference of two x-values
$\dfrac{dy}{dx}$	a notation for the slope of the graph of an equation relating x and y
e	a non-terminating decimal whose digits exhibit no recurring pattern. To four decimal places e = 2.7183
ln	equivalent to $\log_e$
exp	$y = \exp(t)$ is equivalent to $y = e^t$

Introduction

Purpose of This Book and How to Use It

This book is not intended to be a textbook in mathematics that can be read from cover to cover; you should treat it as a reference book.

The aim of the book is to treat, as briefly as possible, the background mathematics assumed in the Science Foundation Course. Each section starts with an introduction which summarizes the results it is going to treat and most end with a few exercises for you to try.

Throughout the Science Foundation Course there will be references to specific sections in this book whose results are being assumed.

No doubt you will treat the material in a variety of ways which might include:

1 You have forgotten a formula (e.g. for the area of a triangle) — a quick reference to the relevant section will give you the formula you want in the introduction, and that may be all you need.

2 You may never have met a formula — again, the relevant section will list the formula in the introduction but you will be able to see how the formula was obtained in the following material and see examples of how it is used. Then there will be exercises for you to try. You might do these immediately or return to them after reading more of the main text.

A glance at the introduction at the front of each section will tell you the range of material that we are assuming and, of course, there is nothing to prevent you working through the parts of the book you are doubtful about before you start on the course itself.

Unless you so desire, there is no reason to refer to this book further until you meet a specific reference in the correspondence text to a mathematical result you have forgotten or not met before.

Section 1 Arithmetic and Algebra

A Indices

Introduction Intro. 1.A

In this section we meet the laws of indices:

$$a^m \times a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^m \times a^{-n} = a^{m-n}$$

$$(a^m)^{1/n} = a^{m/n}$$

$$(a^n)^m = a^{nm}$$

and see examples of indices being manipulated.

1 Products like 1.A.1

1.44×1.44 and $0.02 \times 0.02 \times 0.02$

can be written in the following shorthand notation

$1.44 \times 1.44 = (1.44)^2$ and $0.02 \times 0.02 \times 0.02 = (0.02)^3$

If a is any number and m is a positive integer $(1, 2, 3, \ldots)$, a^m is defined by

$$a^m = \underbrace{a \times a \times a \times \cdots \times a}_{m \text{ factors}}$$

a^m is called the mth power of a and m is called the index or exponent.

2 If n as well as m is a positive integer, the product of powers $a^m a^n$ is 1.A.2

$$a^m \times a^n = \underbrace{(a \times a \times a \times \cdots \times a)}_{m \text{ factors}} \times \underbrace{(a \times a \times a \times \cdots \times a)}_{n \text{ factors}}$$

i.e.

$$a^m a^n = a^{m+n} \qquad (1)$$

Example

$10^3 \times 10^2 = 10^5 = 100\,000$

3 The reciprocal of a^m, $\frac{1}{a^m}$ can be denoted by a^{-m}: for example 1.A.3

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}.$$

The quotient of powers $\frac{a^m}{a^n}$ can be written $a^m a^{-n}$.

Example

$$\frac{3^5}{3^3} = 3^5 3^{-3}$$

Also

$$\frac{3^5}{3^3} = \frac{\cancel{3} \times \cancel{3} \times \cancel{3} \times 3 \times 3}{\cancel{3} \times \cancel{3} \times \cancel{3}} = 3 \times 3 = 3^2$$

So

$$\frac{3^5}{3^3} = 3^5 3^{-3} = 3^2$$

In general

$$\frac{a^m}{a^n} = a^m a^{-n} = a^{m-n} \qquad (2)$$

4 If $m = n$, $\frac{a^m}{a^n} = \frac{a^m}{a^m} = \frac{a \times a \times a \times \cdots \times a}{a \times a \times a \times \cdots \times a}$ 1.A.4

For each a in the denominator there is one in the numerator, and they can all be cancelled.

Therefore

$$\frac{a^m}{a^m} = 1$$

but, by (2)

$$\frac{a^m}{a^m} = a^{m-m} = a^0$$

We therefore define $a^0 = 1$ (3)

5 Since $2 \times 2 \times 2 = 8$, we say 2 is a cube root of 8 and we write 1.A.5

$$(8)^{\frac{1}{3}} = 2$$

For, if (1) is to hold for cases where m and n are numbers other than integers,

$$(8)^{\frac{1}{3}} \times (8)^{\frac{1}{3}} \times (8)^{\frac{1}{3}} = 8^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 8^1 = 8$$

In general, if a is a positive number and q is a positive integer, $a^{1/q}$ is the *positive* qth root of a, i.e.

$$\underbrace{a^{1/q} \times a^{1/q} \times a^{1/q} \times \cdots \times a^{1/q}}_{q \text{ factors}} = a$$

Equivalent notation: $a^{1/q} = \sqrt[q]{a}$ (4)

6 If p is an integer or zero and q is an integer $a^{p/q}$ is defined by 1.A.6

$$a^{p/q} = (a^p)^{1/q} = (a^{1/q})^p \qquad (5)$$

Which form to use in a calculation depends on the numbers involved.

Example

$$16^{\frac{3}{2}} = (16^3)^{\frac{1}{2}} = (16^{\frac{1}{2}})^3$$

Since $16^{\frac{1}{2}}$ is easier to evaluate than 16^3

$$16^{\frac{3}{2}} = (4)^3 = 64$$

7 The mth power of a^n is written $(a^n)^m$ 1.A.7

$$(a^n)^m = \overbrace{\underbrace{(a \times a \times a \times \cdots \times a)}_{n \text{ factors}} \times \underbrace{(a \times a \times a \times \cdots \times a)}_{n \text{ factors}} \times \cdots \times \underbrace{(a \times a \times a \times \cdots \times a)}_{n \text{ factors}}}^{m \text{ factors, each with } n \text{ terms}}$$

$$= \underbrace{a \times a \times a \times \cdots \times a}_{m \times n \text{ factors}}$$

i.e.

$$(a^n)^m = a^{nm} \qquad (6)$$

8 (1), (2), (5) and (6) are often referred to as the laws of indices 1.A.8

$$a^m \times a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^m \times a^{-n} = a^{m-n}$$

$$(a^m)^{1/n} = a^{m/n}$$

$$(a^n)^m = a^{nm}$$

The first and last laws above can be extended to any number of products and powers. For example

$$a^m \times a^n \times a^p \times a^q = a^{m+n+p+q}$$

$$\{[(a^n)^m]^p\}^q = a^{nmpq}$$

Example

$$[(8^2)^{\frac{1}{3}}]^{\frac{1}{2}} = 8^{2 \times \frac{1}{3} \times \frac{1}{2}} = 8^{\frac{1}{3}} = 2$$

9 Division by zero is not defined; so a^{-n} is meaningless when $a = 0$. 1.A.9

Exercise Ex.1.A

1) Evaluate the following.
   (i) $(25)^{\frac{1}{2}}$ (ii) $100^{-\frac{1}{2}}$ (iii) $(a^2)^3/(a^{-4} \times a^{-2})$
2) What is the value of x in the following?
   (i) $\{(a^m)^n\}^p = a^x$ (ii) $b^p \times b^q \times b^r = b^x$ (iii) $a^0 = x$, for all a.

B Logarithms

Introduction Intro. 1.B

This section explains the theory of logarithms and how to read tables of logarithms. Full explanations of calculations using logarithms are presented.

1 *Preliminaries to definition* 1.B.1

i) If a is positive and less than 1, a^n where n is positive or negative is positive
   For example
   $(0.25)^{-2} = 16$ and
   $(8)^{-\frac{1}{3}} = 0.5$
ii) If a equals 1, $a^n = 1$ for all n.
iii) If a is positive and greater than 1, a^n where n is positive or negative is positive.
   For example
   $(1.1)^3 = 1.331$ and
   $(16)^{-\frac{1}{2}} = 0.25$

If x is positive and
$x = a^n$
where a is positive, then n is called the logarithm of x to base a. We write
$n = \log_a x$

2 Here are some examples. 1.B.2

x	a	$\log_a x$	
25	5	2	since $5^2 = 25$
16	2	4	since $2^4 = 16$
16	4	2	since $4^2 = 16$
100	100	1	since $100^1 = 100$
100	10	2	since $10^2 = 100$
64	8	2	since $8^2 = 64$
64	16	1.5	since $16^{\frac{3}{2}} = 64$
0.25	16	−0.5	since $16^{-\frac{1}{2}} = 0.25$

Note that there is no such thing as *the* logarithm of a number; a base must always be specified. Also note that a logarithm can be positive or negative.

Logarithms to base 10, called common logarithms, are used in calculations. You may be able to see why by looking at the laws of indices taken from 1.A.8 with $a = 10$.

Law	*Comment*
$10^m \times 10^n = 10^{m+n}$	multiplication: indices added
$\dfrac{10^m}{10^n} = 10^{m-n}$	division: indices subtracted
$(10^m)^{1/n} = 10^{m/n}$	root of power: indices divided
$(10^m)^n = 10^{mn}$	power of power: indices multiplied

So, for example, to evaluate $\sqrt[5]{1027}$, we first express 1027 in the form 10^m (we find m, the logarithm to base 10 of 1027, by using a set of tables) we divide m by 5 and then use tables again to tell us what number has as

its logarithm to base 10 the number $\frac{m}{5}$. (The use of logarithms in calculations is explained fully in 1.B.7.)

3 Logarithm Form of the Index Laws 1.B.3

Law	*Logarithm form*
$10^m \times 10^n = 10^{m+n}$	$\log_{10}(10^m \times 10^n) = \log_{10} 10^{m+n} = m + n$
$\frac{10^m}{10^n} = 10^{m-n}$	$\log_{10}(10^m/10^n) = \log_{10} 10^{m-n} = m - n$
$(10^m)^{1/n} = 10^{m/n}$	$\log_{10}(10^m)^{1/n} = \log_{10} 10^{m/n} = \frac{m}{n}$
$(10^m)^n = 10^{mn}$	$\log_{10}(10^m)^n = \log_{10} 10^{mn} = mn$

4 To see why logarithm (to base 10) tables are presented in the way they are, we first look at some numbers whose logarithms are easy to find 1.B.4

x	$\log_{10} x$	*Comment*
$\frac{1}{100}$	-2	All numbers between $\frac{1}{100}$ and $\frac{1}{10}$ have logarithms between -2 and -1.
$\frac{1}{10}$	-1	All numbers between $\frac{1}{10}$ and 1 have logarithms between -1 and 0.
1	0	All numbers between 1 and 10 have logarithms between 0 and 1.
10	1	All numbers between 10 and 100 have logarithms between 1 and 2.
100	2	

8 lies between 1 and 10 and hence its logarithm lies between 0 and 1. (In fact it is 0.903 1, correct to 4 decimal places.)

Since any positive number x can be written in the standard form
$x = 10^m \times X$
where m is an integer and $1 \leqslant X < 10$

$$\log_{10} x = \log_{10}(10^m) + \log_{10} X$$
$$= m + \log_{10} X$$

For example
$0.08 = 10^{-2} \times 8$

so
$\log_{10}(0.08) = -2 + 0.903\,1$.

So, to obtain the logarithm of x we write x in the standard form
$x = 10^m \times X$
and add m to $\log_{10} X$, which will be a value between 0 and 1 since X lies between 1 and 10.

To preserve the m part (called the characteristic) and the $\log_{10} X$ part (the mantissa) of $\log_{10} x$, a novel notation is introduced. We write $\bar{2}$ (read as "bar 2") for -2, for example, and write:
$\log_{10}(0.08) = \bar{2}.903\,1$
to mean $-2 + 0.903\,1$ (i.e. $\bar{2}.903\,1$ has the same value as $-1.096\,9$).

Logarithm (to base 10) tables list the logarithms of numbers X between 1 and 10. To write down $\log_{10} x$ we first express x in standard form, i.e.
$x = 10^m \times X$
which determines m, and then use tables to find $\log_{10} X$.

5 Description of Logarithm Tables

This description refers to the logarithm tables on pages 2 and 3 of *New Physical and Mathematical Tables* by Clark (Oliver and Boyd, 1970).

The *main body* of the table (4-digit entries) gives the value of $\log_{10} X$ correct to 4 decimal places when X is a 3-digit number (e.g. 8.25). The right-hand columns under the heading "mean differences" are used to estimate $\log_{10} X$ when X is a 4-digit number: such values can be in error by one or two units in the fourth decimal place. (The use of "mean differences" enables one to short cut the more accurate method of estimating $\log_{10} X$ when X is a 4-digit number, called linear interpolation, but one must sacrifice a little accuracy.)

Example

What is $\log_{10} 0.827\,4$?

First express it in standard form

$0.827\,4 = 10^{-1} \times 8.274$

We next use the table to find $\log_{10} 8.274$.

First locate 82 in the bold face far left column; then locate the column of the main body of the table headed by a bold face 7. The number at the intersection of the 82 row and the 7 column is the value of $\log_{10} 8.27$.

From the table

$\log_{10} 8.27 = 0.917\,5$

(Since 8.27 is between 1 and 10, we know its logarithm is between 0 and 1 even though the table gives no decimal points.) Also

$\log_{10} 8.28 = 0.918\,0$

$\mathrm{Log}_{10}\, 8.274$ will therefore lie between 0.917 5 and 0.918 0. To see how many units in the fourth decimal place to *add* to 0.917 5, locate the number at the intersection of the 82 row and the 4 column of the mean differences, it is 2.

$$\text{Hence } \log_{10} 8.274 = 0.917\,5 + 0.000\,2$$
$$= 0.917\,7$$

Therefore,

$\log_{10} 0.827\,4 = -1 + 0.917\,7$

i.e.

$\log_{10} 0.827\,4 = \bar{1}.917\,7$

The procedure is illustrated below.

											Mean Differences								
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5

6 Antilogarithms

Having performed a calculation using logarithms, e.g. 2.81 ÷ 19.24

x	$\log_{10} x$
2.81	0.448 7
19.24	1.284 2
	$\bar{1}$.164 5, by subtraction

we are left with a number which is the logarithm of the required answer. To find the number we use the tables of antilogarithms given on pages 4 and 5 of *Clark*.

In general, if

$n = \log_{10} x$

$x = \text{antilog}_{10}\, n.$

The tables list n between 0 and 1: the characteristic part of an antilog is easily dealt with since, for example 2.____ implies a factor of 10^2 and $\bar{3}$.____ implies a factor of 10^{-3} in the final number.

Example

Antilogarithm tables are read in exactly the same way as logarithm tables. The extract below shows how the number corresponding to a logarithm of 0.164 5 is found

$\text{antilog}_{10}\, 0.164\,5 = 1.461$

(Note that although no decimal points are given in the table we know that X is between 1 and 10 since its logarithm is between 0 and 1.) Hence,

$$\text{antilog}_{10}\, \bar{1}.164\,5 = 10^{-1} \times 1.461$$
$$= 0.146\,1$$

i.e.

$$2.81 \div 19.24 = 0.146\,1$$

As with logarithm tables there is a possibility of an error of 1 or 2 units in the last decimal place.

											Mean Differences								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3

7 Examples of Calculations

i) Since the characteristic part of a logarithm may be positive or negative and the mantissa is always positive, care must be taken when there is a "carry" from the tenths column to the unit column. In the example, the characteristic 2 is obtained from

x	$\log_{10} x$
0.912 7	$\bar{1}$.960 3
263.1	2.420 2
240.2	2.380 5, by addition
	1

$\bar{1} + 2 + 1 = -1 + 2 + 1 = 2$

$0.912\,7 \times 263.1 = 240.2$

ii) Extracting a root by using logarithms is straightforward when the logarithm of the number has a positive or zero characteristic.

$\sqrt[5]{19.42} = 1.809$

x	$\log_{10} x$
19.42	1.288 2
$\sqrt[5]{19.42}$	0.257 64, division by 5
1.809	0.257 6, correct to 4 decimal places

iii) When the characteristic of a number whose root is required is negative, $(-m)$, we can evaluate the nth root straightforwardly if $-m$ is a whole multiple of n.

Here is an example

$\sqrt[5]{0.000\,012\,34} = 0.104\,3$

x	$\log_{10} x$
0.000 012 34	$\bar{5}.091\,3$
$\sqrt[5]{0.000\,012\,34}$	$\bar{1}.018\,26$, division by 5
0.104 3	$\bar{1}.018\,3$, correct to 4 decimal places.

In such cases the negative characteristic divides exactly and the characteristic and mantissa can be kept separate.

iv) When the characteristic of a number x whose root is required is negative $(-m)$ we can evaluate its nth root by evaluating

$\sqrt[n]{10^n \times x \times 10^{-n}}$

if n is greater than m, and by evaluating

$\sqrt[n]{10^{pn} \times n \times 10^{-pn}}$

if m is greater than n where p is an integer such that pn is greater than m

Here is an example

$\sqrt{0.000\,012\,34}$

x	$\log_{10} x$
0.000 012 34	$\bar{5}.019\,3$

Hence $n = 2$ and $m = 5$
Hence if $p = 3$, pn is greater than m.

$$\sqrt[2]{0.000\,012\,34} = \sqrt[2]{10^6 \times 0.000\,012\,34 \times 10^{-6}}$$
$$\sqrt[2]{12.34 \times 10^6}$$

x	$\log_{10} x$
12.34	1.091 3
10^{-6}	$\bar{6}.000\,0$

We are now in a position to divide the two logarithms by 2 (keeping the mantissa and characteristic separate), the sum of the two resulting logarithms being the logarithm of the root required.

$\sqrt[2]{0.000\,012\,34} = 0.003\,514$

x	$\log_{10} x$
12.34	1.091 3
10^{-6}	$\bar{6}.000\,00$
$\sqrt[2]{12.34}$	0.545 65, division by 2
$\sqrt[2]{10^{-6}}$	$\bar{3}.000\,0$, division by 2
$\sqrt[2]{12.34 \times 10^{-6}}$	$\bar{3}.545\,65$, addition
0.003 514	$\bar{3}.545\,7$, correct to 4 decimal places.

8 A Note on Accuracy 1.B.8

Since readings from both logarithm and antilogarithm tables may be in error by a unit or two in the last decimal place, answers to a calculation can only be counted accurate to 3 significant figures. Hence the answers to the examples in 1.B.7 should be

i) 240 ii) 1.81 iii) 0.104 iv) 0.003 51

9 Since the logarithm of a negative number is not defined, a calculation involving negative numbers is treated in the following way. 1.B.9

Example

Evaluate

$-1.23 \times -1.94 \times -2.87$

Since three factors -1 are involved, the answer is

$-1 \times (1.23 \times 1.94 \times 2.87)$

The product in brackets can be found by using logarithms since all the numbers are positive.

Ex.1 B

Exercise

Use logarithm and antilogarithm tables to evaluate

i) $\dfrac{1.891 \times \sqrt[3]{0.121\,2}}{(7.89)^{-3}}$

ii) $(7.27)^3 \times (0.827\,1)^{-4}$

iii) $(0.21)^3 + (0.31)^2 + (0.41)^2$

iv) $(1.414)^4$

v) $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

C A Miscellany of Notation and Technique

Introduction Intro. 1.C

This section discusses simplifying algebraic expressions by using the laws of indices and the expansions

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$
$$(a + b)(a - b) = a^2 - b^2$$

Also introduced are modulus, direct proportion, the Σ notation for sums, and an approximation to $(a + b)^2$.

1 An algebraic expression can be something as simple as x or as complicated as 1.C.1

$$5\left(\frac{ax^2 + 2by}{ax - 3cy^2}\right)$$

Sometimes it is useful to use the word *term* to describe one grouping of symbols; thus, in the above example, ax^2, $2by$, ax and $3cy^2$ might be referred to as terms

2 The use of indices to simplify the presentation of an algebraic expression is facilitated by the use of parentheses. 1.C.2

The product of $a + b$ with $c + d$ is written

$(a + b)(c + d)$

and the product of $a + b$ with itself is written

$(a + b)^2$

If we wish, we can expand $(a + b)^2$ by writing out all the products of the terms

$$\begin{aligned}(a + b)^2 = (a + b)(a + b) &= a(a + b) + b(a + b)\\ &= a^2 + ab + ba + b^2\end{aligned}$$

i.e. $$(a + b)^2 = a^2 + 2ab + b^2 \qquad (1)$$

Similarly $$\begin{aligned}(a - b)^2 = (a - b)(a - b) &= a(a - b) - b(a - b)\\ &= a^2 - ab - ba + b^2\end{aligned}$$

$$\text{i.e.} \qquad (a - b)^2 = a^2 - 2ab + b^2 \qquad (2)$$

$$\text{Also} \qquad (a - b)(a + b) = a(a + b) - b(a + b)$$
$$= a^2 + ab - ba - b^2$$

$$\text{i.e.} \qquad (a - b)(a + b) = a^2 - b^2 \qquad (3)$$

(1), (2) and (3) are used so often that one should really feel happy with them.

Examples

i) $(3x - 2y)(3x + 2y) = 9x^2 - 4y^2$, by (3).

ii) $(2z^2 - x)^2 = (2z^2)^2 - 2(2z^2)x + x^2$, by (2)
$= 4z^4 - 4z^2x + x^2$

iii) $(x - y)^2 - (x + y)^2 = x^2 - 2xy + y^2 - (x^2 + 2xy + y^2)$, by (2) and (1)
$= -4xy$.

Consider (1) for the case $a = 10$, $b = 0.1$
We have

$$(a + b)^2 = a^2 + 2ab + b^2$$

so

$$(10 + 0.1)^2 = 10^2 + 2 \times 10 \times 0.1 + (0.1)^2$$
$$= 100 + 2 + 0.01$$

In this case the b^2 term is very small when compared with either of the other terms. In fact 102 is a very good approximation to $(10.1)^2$.

Whenever b is very small compared with a, we can approximate $(a + b)^2$ by $a^2 + 2ab$; we write this as
$(a + b)^2 \approx a^2 + 2ab$, if $b \ll a$
where $\ll$ means "is very much smaller than".

Here are two more examples.

i) $(1000 + 1)^2 \approx 10^6 + 2 \times 10^3$

ii) $(10 + 5)^2 = 225$
$10^2 + 2 \times 10 \times 5 = 200$

In this case $(a + b)^2$ is not well approximated by $a^2 + 2ab$ since 5 is not "very much smaller than" 10

3 Complicated algebraic expressions can sometimes be simplified by applying the laws of indices and the expansions (1), (2) and (3) above. 1.C.3

Examples

i) Simplify

$$\frac{(a + b)^2 - [(a - b)^4]^{\frac{1}{2}}}{4ab}$$

Applying the power of a power law

$$\frac{(a + b)^2 - [(a - b)^4]^{\frac{1}{2}}}{4ab} = \frac{(a + b)^2 - (a - b)^2}{4ab}$$

Expand the right-hand side by using (1) and (2)

$$= \frac{a^2 + 2ab + b^2 - (a^2 - 2ab + b^2)}{4ab}$$

Cancel terms

$$= \frac{4ab}{4ab}$$

i.e.

$$\frac{(a + b)^2 - [(a - b)^4]^{\frac{1}{2}}}{4ab} = 1$$

ii) Simplify

$$\frac{(xyz^2)^3}{\left(\frac{1}{x}y^{-2}\right)^2}$$

Since

$$a^{-n} = \frac{1}{a^n}$$

$$\frac{(xyz^2)^3}{\left(\frac{1}{x}y^{-2}\right)^2} = \frac{(xyz^2)^3}{(x^{-1}y^{-2})^2}$$

Apply the power of a power law to the numerator and denominator

[illegible]

Again, use $a^{-n} = \frac{1}{a^n}$

[illegible]

i.e.

$$\frac{(xyz^2)^3}{\left(\frac{1}{x}y^{-2}\right)^2} = [illegible]$$

4 If the letter d represents a physical quantity which can be positive or negative and we are only interested in the magnitude not the sign of that quantity, we denote this by the symbol 1.C.4

$|d|$

read as "mod d", short for modulus of d.

Examples

[illegible]

In general

$|d| = |-d|$

Note that

$|0| = 0$

Examples

i) If $a = 6$ and $b = -7$, evaluate $|a + b|$ and $|a| + |b|$

$|a + b| = |6 - 7| = |-1| = 1$

$|a| + |b| = |6| + |-7| = 6 + 7 = 13$

ii) If $a = 3$ and $b = 5$, evaluate $|a + b|$ and $|a| + |b|$

$|a + b| = |3 + 5| = |8| = 8$

$|a| + |b| = |3| + |5| = 3 + 5 = 8$

These two examples illustrate the general rule that

$|a + b| \leqslant |a| + |b|$

i.e. the modulus of a sum is less than or equal to the sum of the moduli

5 Suppose that in an experiment you collected a series of 10 time readings: 1.C.5
$t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}$.

The mean or average time reading is

$$\frac{t_1 + t_2 + t_3 + \dots + t_{10}}{10}$$

A sum like that in the numerator is conveniently represented by

$$\sum_{i=1}^{10} t_i$$

i.e.

$$\sum_{i=1}^{10} t_i = t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7 + t_8 + t_9 + t_{10}$$

The symbol, Σ is the Greek letter sigma and $\sum_{i=1}^{10} t_i$ is read as "the sum of t_i from i equals 1 up to 10." Any letter can be used in this way, but it always takes on a run of consecutive integer values. Other examples are

$$\sum_{r=3}^{9} r = 3 + 4 + 5 + 6 + 7 + 8 + 9$$

and

$$\sum_{k=1}^{4} (k-1)^2 = (1-1)^2 + (2-1)^2 + (3-1)^2 + (4-1)^2$$

For a series of n time readings we can define the mean time reading, $\bar{t}$, (read as "t bar") by

$$\bar{t} = \frac{1}{n}\left[\sum_{i=1}^{n} t_i\right]$$

6 Suppose t and s represent some physical quantities and that from an experiment or series of experiments a table of corresponding values of t and s are gathered I C 6

t	s
1.7	5.0
2.9	8.7
3.8	11.4
4.6	14.7
5.9	17.7

By inspecting the table we can see that the ratio

$$\frac{s \text{ value}}{t \text{ value}}$$

is approximately 3 in each case.

On the basis of these values we might conclude that there is an underlying physical truth: the ratio of the s value to the t value is always the same

In such a case we say s and t are in direct proportion; in symbols

$s \propto t$

From this one can write

$$\frac{s}{t} = k$$

where k is some constant.

In the example above, we could take $k = 3$, then

$$\frac{s}{t} = 3$$

or

$$\frac{t}{s} = \frac{1}{3}$$

To confirm this equation we could observe s for more values of t between 1.7 and 5.9; we would also want to observe s for values of t outside this range.

See also III.C, where another example of direct proportion is discussed

Exercise Ex I C

1. i) $(X - Y)^2$ ii) $(X + Y)(X - Y)$
   iii) $(3a + b)^2$ iv) $(2a + c^2)^2(c^2 - 2a)$

2. Show that
$$\frac{(x - a - b)(a + b + x) + a^2 + 2ab + b^2}{x^2} = 1$$

3. If $x \ll y$, show that $(x + y)^2 \approx y(y + 2x)$.

4. Simplify

i) $\dfrac{(x^2 y z^2)}{(x^{-2} y\, z^{-2})}$ ii) $[(x + 2y)^2 - (2x + y)^2]^2$

5. Evaluate $|a - b|$ where $a = -3$ and $b = -4$.

6. Evaluate $\sum_{j=2}^{4} 2^j$

7. Can one deduce that x and y are in direct proportion from the following tables?

i)

x	y
1	1
2	4
3	9
4	16

ii)

x	y
1.4	3.5
2.4	6.0
3.4	8.5
6.4	16.0

iii)

x	y
1	1.00
2	0.50
3	0.33
5	0.20

D Manipulating Equations and Formulae

Introduction Intro. 1.D

This section explains what an equation is and gives guidelines for manipulating them by performing allowable operations.

1 A **variable** is a quantity which can take on any of the numbers of some set of numbers; the set of numbers might be, for example, all numbers, only the positive integers, or all the numbers between -4 and $+6$. 1.D.1

A variable is usually represented by a letter from the last part of the alphabet, although no harm is done, of course, if we wish to use the initial letter of a physical quantity which is the variable under discussion.

An **equation** is a statement that two expressions, one of which involves at least one variable, are equal. (In what follows we assume that the variable can represent any number of the set of all numbers, unless otherwise stated.)

The following, in which x and y are variables, are all equations.

$$x + y = 1 \tag{1}$$

$$2x = y \tag{2}$$

$$x + 1 = 2 \tag{3}$$

$$x = x \tag{4}$$

The equation

$v = u + ft$

where

v represents the speed of a body after time t

u represents the speed of the body at some starting time, $t = 0$, say

and

f is the constant acceleration of the body

is said to be a **formula** for v in terms of u, f and t.

If we subtract 1 from both sides of (3) we obtain

$x + 1 - 1 = 2 - 1$

i.e. $x = 1$

"$x = 1$" is the **solution** of (3).

In part E: Solving Equations, we explain how to find the solution of, i.e. solve, certain types of equation.

2 When we subtracted 1 from both sides of (3), we were performing an allowable operation on an equation. Allowable operations enable us to manipulate equations in such a way that the outcome is another equation 1.D.2

whose solution is the same as the original equation: such equations are called equivalent equations. In the examples which follow the allowable operations are printed in bold.

3 *Example 1* 1.D.3

If F is the temperature in degrees Fahrenheit and C is the temperature in degrees Celcius, then

$$F - 32 = \tfrac{9}{5}C \quad (5)$$

What temperature on the Fahrenheit scale corresponds to 10 on the Celcius scale?

Substitute 10 for C in (5)

$$F - 32 = \tfrac{9}{5} \cdot 10$$

(Note that the dot on the right-hand side denotes multiplication.)

We can simplify the right-hand side to obtain

$$F - 32 = 18$$

Add to both sides 32

$$F - 32 + 32 = 18 + 32$$

i.e.

$$F = 50$$

Example 2

If we knew the value of F and required to find C, we could make C the subject of the formula in the following way.

Multiply both sides of (5) by $\tfrac{5}{9}$

$$\tfrac{5}{9}(F - 32) = \tfrac{5}{9} \cdot \tfrac{9}{5}C$$

If we now change the order of the sides and simplify we have

$$C = \tfrac{5}{9}(F - 32)$$

Example 3

Make u the subject of the equation

$$puv = \frac{\psi v}{u} \quad (6)$$

Multiply both sides of (6) by u

$$pu^2v = \psi v \quad (7)$$

Divide both sides of (7) by pv

$$u^2 = \frac{\psi \cancel{v}}{p\cancel{v}} \quad (8)$$

Cancel v in numerator and denominator of right-hand side and then **take the square root** (positive and negative) of both sides of (8)

$$u = +\sqrt{\frac{\psi}{P}} \quad \text{and} \quad u = -\sqrt{\frac{\psi}{P}}$$

This can be written

$$u = \pm\sqrt{\frac{\psi}{P}}$$

(Note that if u, ψ and p represent physical quantities it might be possible to rule out one of the signs, + or −, from physical considerations.)

Example 4

Consider two equations which are true at the same time

$$y = ax + b \quad (9)$$

and

$$Y = AX + B \quad (10)$$

The allowable operations indicated above can be used to produce just one equation from (9) and (10). Since $Y = AX + B$, we can perform the

allowable operation **divide both sides of (9) by Y** by dividing its left-hand side by Y and the right-hand side by $AX + B$.

Hence

$$\frac{y}{Y} = \frac{ax + b}{AX + B}$$

Exercise Ex.1.D

1. Make u the subject of the formula
   $v^2 = u^2 + 2fs$
   by performing allowable operations.

2. $v = \pi r^2 h$
   $V = \pi R^2 H$
   $H = 2h$
   By performing allowable operations, express $\frac{v}{V}$ in terms of r and R only.

3. A cylinder of length h and radius r encloses a volume V_1 given by
   $V_1 = \pi r^2 h$.
   A sphere of radius r encloses a volume V_2 given by
   $V_2 = \frac{4}{3}\pi r^3$
   If $V_2 = 2V_1$, express h in terms of r by performing allowable operations.

E Solving Equations

Introduction Intro. 1.E

In this section we solve

linear equations $ax + b = 0$

quadratic equations $ax^2 + bx + c = 0$

simple cubic equations $ax^3 + b = c$

Quadratic equations are solved by means of the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The method of elimination is used to solve pairs of simultaneous equations in two variables

1 In 1.D.1 we saw the solution of an equation being obtained by performing allowable operations. 1.E.1

To solve an equation we perform allowable operations until we reach an equivalent equation of the form

$x =$ something known

which is the solution of the original equation.

2 *Example 1: Linear Equation in x* 1.E.2

Solve $2x + 3 = -5$ (1)

Subtract 3 from both sides of (1)

$2x + 3 - 3 = -5 - 3$

i.e.

$2x = 8$ (2)

Divide both sides of (2) by 2

$x = -4$ (3)

(3) is the solution of (1).

In general

$ax + b = c$

in which a is not zero has solution

$$x = \frac{c - b}{a}$$

A solution can always be checked by substitution.

Substitute

$x = -4$ in (1)

The left-hand side becomes: $2 \cdot (-4) + 3 = -5$, as required.

3 *Example 2: Equations with No Solution* I.E.3

Solve $x^2 = -4$

We cannot take the square root of both sides of this equation since there is no ordinary number whose square is a negative number: the equation has no solution

Example 3: Quadratic Equation in x

Solve $2x^2 + x - 2 = 4$ (4)

Equations like this can always be solved comparing the equivalent equation

$2x^2 + x - 6 = 0$ (5)

with the general quadratic equation in x, for which $a \neq 0$,

$ax^2 + bx + c = 0$ (6)

whose solutions are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The formula for the solution of (6) is generally written

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (7)$$

We now solve (5) by using (7).

By comparing (5) and (6), $a = 2$, $b = 1$, and $c = -6$.

Substitute these values in (7)

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 2(-6)}}{2.2}$$

i.e.

$$x = \frac{-1 \pm \sqrt{49}}{4}$$

So

$x = -2$ and $x = \frac{3}{2}$

Again the solution can be checked by substitution. When $x = -2$, the left-hand side of (5) becomes

$2 \cdot (-2)^2 + (-2) - 6 = 0$, as required.

When $x = \frac{3}{2}$, the left-hand side of (5) becomes:

$2 \cdot (\frac{3}{2})^2 + (\frac{3}{2}) - 6 = 4\frac{1}{2} + 1\frac{1}{2} - 6 = 0$ as required.

In Exercise I.E.2 you will be asked to justify the formula (7).

Note that

i) if $b^2 - 4ac$ is positive, (6) has two solutions.

ii) if $b^2 - 4ac$ is zero, (6) has one solution.

iii) if $b^2 - 4ac$ is negative, (6) has no solutions that are ordinary numbers since no ordinary number is the square root of a negative number.

4 *Example 4 Simple Cubic Equation in x* 1.E.4

Solve

$2x^3 + 4 = -12$ **(8)**

Subtract 4 from both sides of (8)

$2x^3 + 4 - 4 = -12 - 4$

i.e.

$2x^3 = -16$ **(9)**

Divide both sides of (9) by 2

$x^3 = -8$ **(10)**

Take the cube root of both sides of (10)

$x = (-8)^{\frac{1}{3}}$

i.e.

$x = -2$

The solution can be checked by substitution. When $x = -2$, the left-hand side of (8) becomes

$2(-2)^3 + 4 = -12$, as required.

In general, the simple cubic equation, for which $a \neq 0$,

$ax^3 + b = c$

has solution

$$x = \left(\frac{c - b}{a}\right)^{\frac{1}{3}}$$

If $c - b$ is positive, x is positive; if $c - b$ is negative, x is negative.

In practice, the calculation of the cube root of $\dfrac{c - b}{a}$ may have to be performed by using logarithms.

5 Simultaneous Equations 1.E.5

Examples 1, 3 and 4 each involved one equation in one variable. We next consider two equations which each involve the same two variables. For example

$2x + 4y = 3$

$x - 2y = 1$

Such equations are called simultaneous equations.

By performing allowable operations on a pair of simultaneous equations we aim to reach two equivalent equations of the form

$x =$ something known

$y =$ something known

which form the solution of the pair of simultaneous equations.

Example

Solve

$2x + 4y = 19$ (1)

$x - y = -1$ (2)

We will solve for x first. Since we require

$x =$ something known

we manipulate (1) and (2) so as to eliminate y. We can do this, for example, by multiplying both sides of (2) by 4 and then adding the resulting equation to (1). (2) becomes

$4x - 4y = -4$ (3)

add (1) and (3)

$6x + 0 = 19 - 4$

$x = 2.5$ (4)

To obtain y, substitute the x-value in the last equation which involved y, (3)

$4 \times (2.5) - 4y = -4$

$$y = 3.5 \qquad (5)$$

(4) and (5) give the solution of (1) and (2).

The above is called the method of elimination. The solutions can be checked by substitution in (1) and (2).

Check:

the left-hand side of (1) becomes

$2 \times (2.5) + 4 \times (3.5) = 19$, as required.

the left-hand side of (2) becomes

$2.5 - 3.5 = -1$, as required.

6 Eliminating a variable can be a useful technique when deriving a formula. I.E.6

Example

A body of mass m and velocity v has momentum p and kinetic energy E given by

$$p = mv \qquad (1)$$

and

$$E = \tfrac{1}{2}mv^2 \qquad (2)$$

Express E in terms of p and m.

Multiply the left-hand side of (1) by itself and the right-hand side of (1) by itself

$$p^2 = m^2v^2 \qquad (3)$$

Divide the left-hand side of (2) by the left-hand side of (3), and the right-hand side of (2) by the right-hand side of (3)

$$\frac{E}{p^2} = \frac{\frac{1}{2}mv^2}{m^2v^2} \qquad (4)$$

Cancel the v^2 term and one m and then multiply both sides by p^2

$$E = \frac{p^2}{2m}$$

7 In practice when one manipulates or solves an equation the description of the allowable operations being performed is compressed and sometimes two are performed in one go. This can increase the likelihood of making an error; hence one should *always* check a solution where possible. I.E.7

Exercise Ex.I.E

1. Solve the following equations by performing allowable operations. (Remember to check your solutions.)
   i) $9x + 8 = -4$
   ii) $6x^2 + 7x - 3 = 0$
   iii) $9x^2 + 18x + 8 = 0$
   iv) $6x^3 = 10^{-6}$ (use logarithms)
   v) $2x - y = 3$
      $x + 3y = -2$
   vi) $x + y = 3$
      $6x - 2y = -2$

2. Check that $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ is a solution of $ax^2 + bx + c = 0$ by substituting it into the left-hand side of the equation and showing that it reduces to zero.

3. If
   $v^2 = u^2 + 2fs$
   and
   $v = u + ft$
   show that $ft^2 + 2ut - 2s = 0$ by eliminating v.

Answers to Exercises for Section 1

Ex 1A

1. i) 125 ii) $\frac{1}{10} = 0.1$ iii) a^{12}
2. i) mnp ii) $p + q + r$ iii) 1

Ex 1B

i) 459.5 = 460 (to 3 significant figures)
ii) 409.3 = 409 (to 3 significant figures)
iii) 0.555 2 = 0.555 (to 3 significant figures)
iv) 3.996 = 4.00 (to 3 significant figures)
v) 40 330 = 40 300 (to 3 significant figures)

Ex 1C

1. i) $X^2 - 2XY + Y^2$ ii) $X^2 - Y^2$
   iii) $9a^2 + 6ab + b^2$ iv) $c^6 + 2ac^4 - 4a^2c^2 - 8a^3$
4. i) 1 ii) $9y^4 - 18x^2y^2 + 9x^4$
5. 1
6. $2^2 + 2^3 + 2^4 = 28$
7. i) NO ii) YES iii) NO

Ex 1D

1. [illegible] $2h$
2. [illegible] $2R$ [illegible]
3. h [illegible]

Ex 1E

1. i) [illegible]
   ii) [illegible] $-\frac{3}{2}$ and $x = \frac{1}{3}$
   iii) $x = -\frac{4}{3}$ and $x = \frac{2}{3}$
   iv) 0.005 50 (to 3 significant figures)
   v) $x = 1$ and $y =$ [illegible]1
   vi) $x = \frac{1}{2}$ and $y = 2$.

Section 2 Geometry

A The Area of a Triangle

Introduction Intro. 2.A

In this section we establish the formula

$\frac{1}{2} \times$ base $\times$ height

for the area of a triangle

1 A triangle is a closed three-sided plane figure. If two of the sides of a triangle are perpendicular, i.e. are at right-angles, we say that the triangle is right-angled. (Fig. 1.) 2.A.1

The sum of the interior angles of a triangle is two right-angles. Therefore, from Fig. 2

$\alpha + \beta + \gamma = 2$ right angles

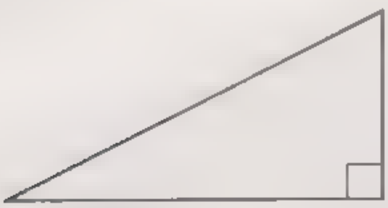


Fig. 1

Fig. 2

2 To determine the area of a triangle ABC we construct a rectangle $ABDE$ on the triangle using one of its sides as one of the sides of the rectangle (Fig. 3). CF is perpendicular to AB. 2.A.2

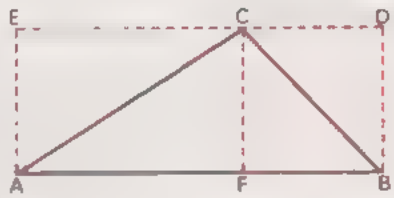


Fig. 3

We can regard AB as the base of triangle ABC; we denote its length by b.

CF is the height of triangle ABC which we may represent by h.

Hence

$AB = ED = b$

and

$BD = AE = FC = h$

Since the diagonals AC and BC divide in half the rectangles $AFCE$ and $FBDC$, respectively,

Area AFC = Area ACE

and

Area FBC = Area CBD

So, since

area AFC + area FBC + area ACE + area CBD = area $ABDE$

and

$$\text{area } AFC + \text{area } FBC = \text{area } ABC$$

$$2 \times \text{area } ABC = \text{area } ABDE$$

But

$$\text{area } ABDE = b \times h$$

Hence

$$\text{area } ABC = \tfrac{1}{2}b \times h$$

In words: the area of a triangle is equal to the product

$\frac{1}{2} \times$ base $\times$ height

Example

The area of triangle XYZ in Fig. 4 is

$\frac{1}{2}$ × base × height
$= \frac{1}{2}(4 \text{ units}) \times (3 \text{ units})$
$= 6$ square units.

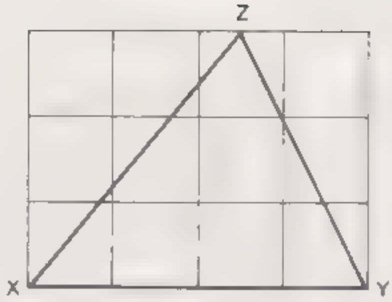


Fig 4

B Area of a Parallelogram

Introduction Intro. 2.B

In this section we establish the formula for the area of a parallelogram and show that two parallelograms between a pair of parallel lines whose bases are equal, have the same area.

1 A quadrilateral is a closed four-sided plane figure. Then, as you know, a rectangle is a quadrilateral each of whose interior angles is a right angle: a square is a rectangle whose sides are each of the same length. A parallelogram is a quadrilateral both of whose opposite pairs of sides are parallel. (Thus, a rectangle is a special case of a parallelogram.) 2.B.1

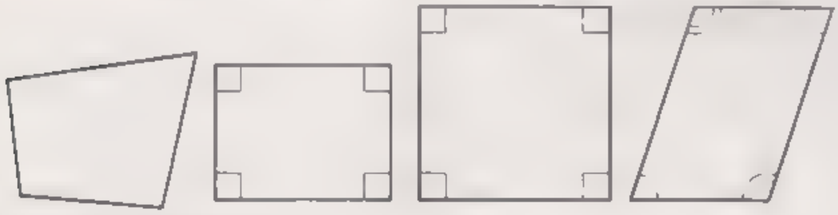


Fig 1 Some Quadrilaterals

By dividing a quadrilateral into two triangles the area of a quadrilateral can be found (Fig. 2)

We can give no formula for the area of any quadrilateral but we will deduce the formula for the area of a parallelogram.

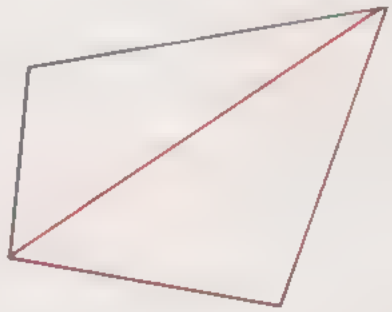


Fig 2

2 In Fig. 3, a parallelogram is shown with one pair of its parallel sides produced. 2.B.2

The diagonal AC divides the parallelogram exactly in two. i.e., area ABC = area DAC

and, therefore,

area $ABCD = 2 \times$ area ABC

But

area $ABC = \frac{1}{2}$base × height
$= \frac{1}{2}AB \times h$

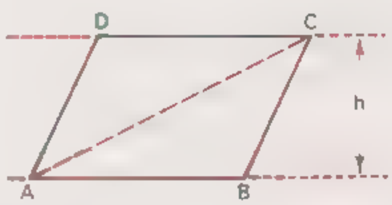


Fig 3

where h is the perpendicular distance between the extended parallel sides.

Therefore

area $ABCD = 2 \times (\frac{1}{2}AB \times h)$

$= AB \times h$

In words:

the area of a parallelogram is equal to the product of one of its sides and the perpendicular distance between that side and the side parallel to it.

Example

The area of the parallelogram $ABCD$ in Fig. 4 is

$AB \times h$

$= 6$ units $\times$ 4 units

$= 24$ square units

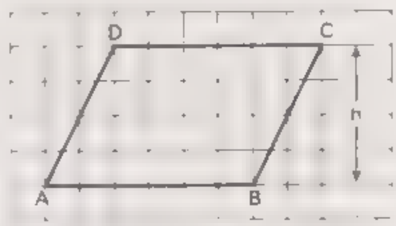


Fig. 4

3 Fig. 5 shows two parallelograms between a pair of parallel lines. **2.B.3**

$AB = XY$

and the perpendicular distance between the parallel lines is h.

Area $ABCD = AB \times h$

Area $XYZW = XY \times h$

But

$AB = XY$

Hence

Area $ABCD$ = Area $XYZW$

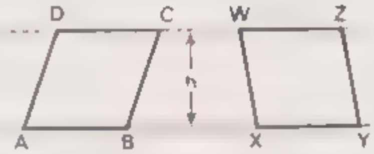


Fig. 5

If we call the side of a parallelogram between the extended parallel lines its base we can express the above results in words:

Two parallelograms between a pair of parallel lines whose bases are equal, have the same area.

Example

Why are parallelograms $ABCD$ and $ABEF$ equal in area? (Fig. 6.)
Since they have a common base, AB, they have an equal base. Hence, from the result above they have equal areas.

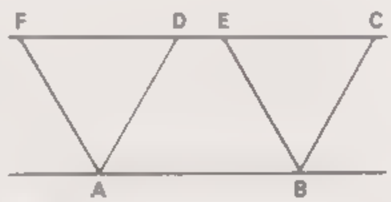


Fig. 6

C Some Geometry of the Circle

Introduction **Intro. 2.C**

In this section we meet the formulae for the circumference of a circle

circumference $= 2\pi r$

and for the area of a circle

area $= \pi r^2$

We also meet the words: arc, chord, tangent, sector and segment.

1 A circle is the set of *all* points in a plane which are at a fixed distance from a fixed point. 2.C.1

The fixed point is called the centre of the circle. Any straight line joining the centre to a point on the circle is called a radius.

Fig 1

A straight line drawn from any point in the circle through the centre to another point on the circle is a diameter of the circle. If d represents the length of a diameter and r the length of the radius, then

$d = 2r$

The distance from any point on a circle round the circle and back to that point is called the circumference of the circle. With the aid of string and cylinders of circular cross-section, it can be shown that the ratio

$$\frac{\text{circumference}}{\text{diameter}}$$

is approximately equal to 3.14

This ratio, in fact, is constant for all circles and is denoted always by the Greek letter π (pi). π is a never-ending decimal which exhibits no recurring pattern in its digits. For most work, the approximation

$\pi = 3.14$ (accurate to 2 decimal places)

is sufficient. As a rational fraction π can be approximated by $3\frac{1}{7}$.

We have, therefore,

circumference $= \pi d = 2\pi r$

2 The area A enclosed by a circle can be shown to be given by 2.C.2

$A = \pi r^2$

(You might like to estimate the area of the circle shown in Fig. 2 whose radius is 10 units.)

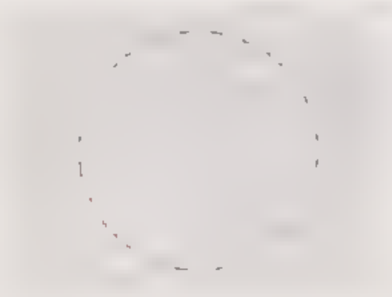


Fig 2

Example

A circle A has an area four times that of a circle B. How many times is the circumference of circle A greater than that of circle B?

If A has radius r and B has radius R

area of $A = \pi r^2$

area of $B = \pi R^2$

and

$$\pi r^2 = 4\pi R^2$$

Hence, since r and R are positive

$r = 2R$

Circumference of $A = 2\pi r$

Circumference of $B = 2\pi R = 4\pi r$

Hence, circumference of A is *twice* that of B.

3 A connected part of a circle is called an arc. In Fig. 3 a circle is shown divided into two arcs. 2.C.3

A straight line drawn from one end of an arc to the other is called a chord.

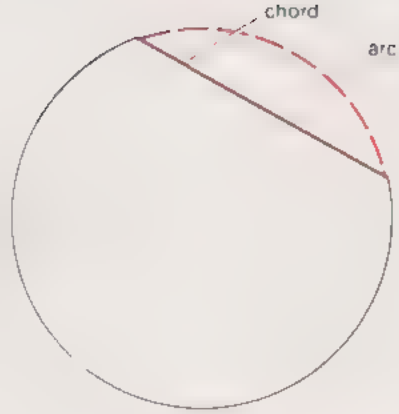


Fig 3

4 The radius which bisects a chord does so at right angles. In Fig. 4 2.C.4

$PR = RQ$

Also this radius bisects the arc:

arc PT = arc TQ

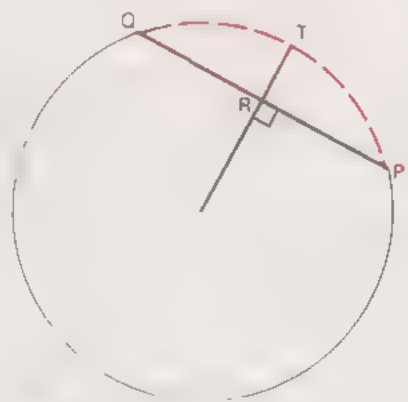


Fig 4

5 Imagine now, reducing the arc PQ in such a way that 2.C.5

arc PT = arc TQ

Eventually, the chord PQ will dwindle to the point T. In Fig. 5 for successive arcs P_1Q_1, P_2Q_2, $P_3Q_3, \ldots$, we have extended the chord beyond the circle by straight lines. Each of those lines is perpendicular to the radius. In the end the extended version of the chord becomes a straight line perpendicular to the radius at T which touches the circle at T only. Such a straight line is called a tangent.

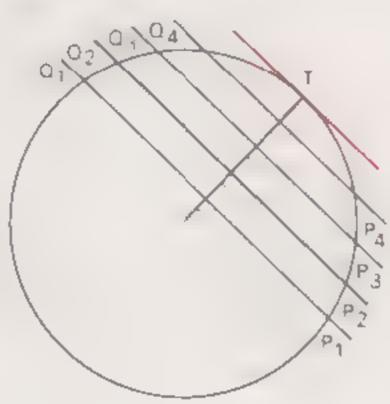


Fig 5

6 Note that arc PQ is longer than chord PQ. Also a diameter is a special case of a chord; in this case the arc is a semi-circle and its length is πr. 2.C.6

7 A segment of circle is the region enclosed by a chord PQ and an arc PQ (Fig. 6) 2.C.7

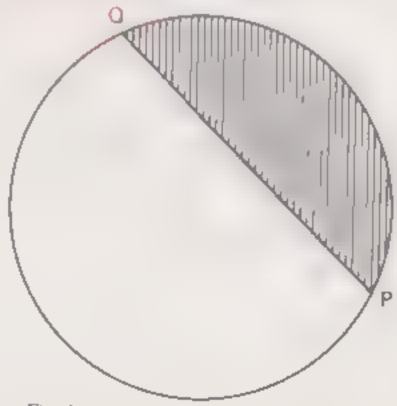


Fig 6

A sector of a circle is the region enclosed by two radii to points P and Q and the arc PQ (Fig. 7)

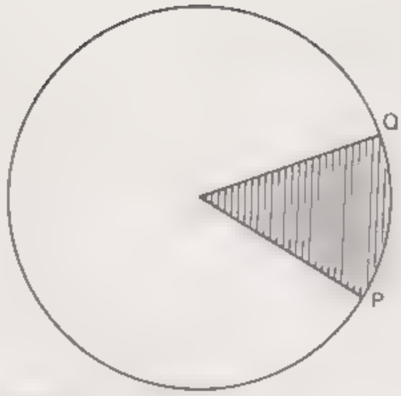


Fig. 7

Example

Why are the tangents at either end of a semi-circle parallel? (Fig. 8) T_1T_2 is a chord which passes through C, the centre of the circle; i.e. T_1CT_2 is a diameter. T_1CT_2 is perpendicular to both tangents since CT_1 and CT_2 are radii and hence the tangents are parallel.

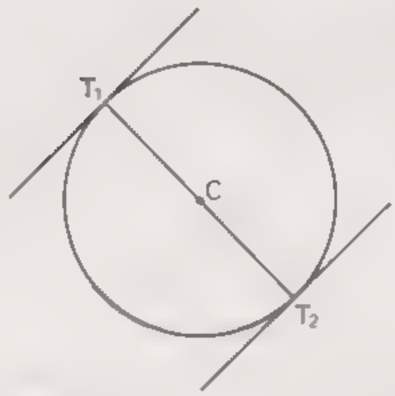
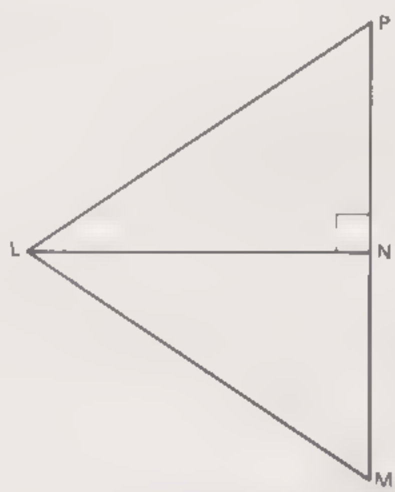


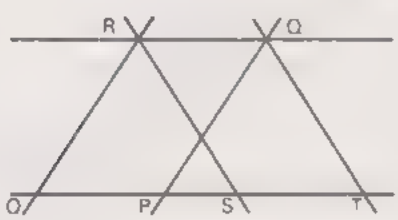
Fig. 8

Exercise Ex.2 ABC

1. What is the area of triangle LMP if PM = 4 in. and LN = 3 in?



2. In the figure: OR and PQ are parallel; SR and TQ are parallel and RQ and OT are parallel.
   i) Why are the areas of $OPQR$ and $STQR$ equal?
   ii) If RQ is 4 in and the perpendicular distance from RQ to OT is 5 in, what is the area of $OPQR$?

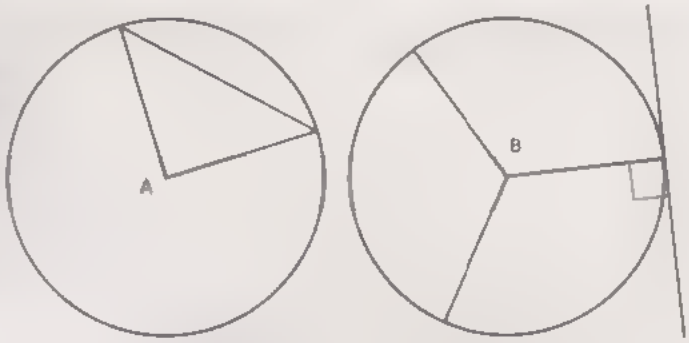


3. If a circle has radius 3 inches, calculate to one decimal place
   i) its area
   ii) its circumference

4. If the diameter of a circle is trebled in size, what is the effect on the area of the circle?

5. On the two given circles with centres at A and B, label the following features:

an arc
a chord
a tangent
a radius
a sector
a segment



D Units of Angle

Introduction Intro. 2.D

This section defines the units of angle, the radian and the degree, and establishes the equation relating them

2π radians = 360 degrees

1 If the lines OA and OB are initially coincident and then OA is rotated anticlockwise from OB about O to a new position, we say OA has passed through an angle; this angle is shown as θ in Fig. 1. 2.D.1

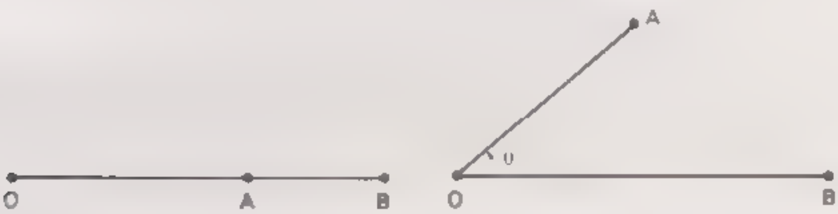


Fig 1

2 There are various *units of angle*, just as there are for any physical quantity. (In fact, we have already used one: the right angle.) If $OA = OB = r$ units of length, and the arc of circle $AB = s$ units of length, then the size of the angle θ in radians is $\frac{s}{r}$. (Fig. 2) The abbreviation rad is used; i.e. 2.D.2

$\theta = \frac{s}{r}$ rad.

When

$s = r$,

$\theta = 1$ rad.

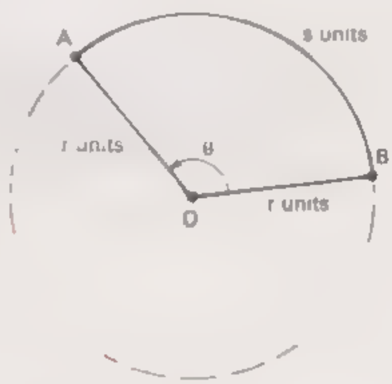


Fig 2

If OA rotates through the full circle, the angle 1 revolution is swept out.

Since the circumference of a circle is $2\pi \times$ radius

$$1 \text{ rev} = \frac{2\pi r}{r} \text{ rad}$$

$$\text{i.e. } 1 \text{ rev} = 2\pi \text{ rad} \qquad (1)$$

A common unit of angle is the degree. In 1 revolution there are 360 degrees. Since the abbreviation for degree is °

1 rev = 360° (2)

and from (1) and (2)

2π rad = 360° (3)

From (3)

$$1 \text{ rad} = \left(\frac{360}{2\pi}\right)^\circ$$

i.e. 1 radian is just less than 60°, since $\pi = 3.14$ to 2 decimal places.

3 The following table shows some useful equivalences between the degree and radian measure of an angle. 2.D.3

Radians	*Degrees*
0	0
$\frac{\pi}{6}$	30
$\frac{\pi}{4}$	45
$\frac{\pi}{3}$	60
$\frac{\pi}{2}$	90
π	180
$\frac{3\pi}{2}$	270
2π	360

E Similar and Congruent Triangles

Introduction Intro. 2.E

In this section we define similar triangles and congruent triangles. Also, criteria for similarity and congruency are stated. A property of similar triangles which enables one to make deductions about the lengths of sides of similar triangles is also discussed.

1 In Fig. 1, the two triangles ABC and ADE have the same shape 2.E.1

i.e.

$\angle ABC = \angle ADE$

$\angle BCA = \angle DEA$

and

$\angle A$ is common to both.

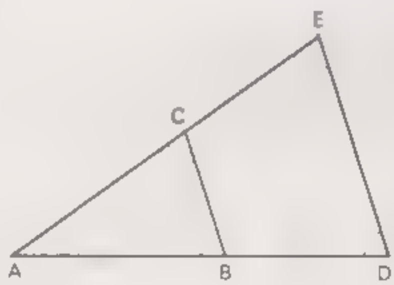


Fig. 1

Triangle ADE can be regarded as a magnification of triangle ABC.

Two triangles are said to be similar if the angles of one are α, β and γ and the angles of the other are also α, β and γ.

In Fig. 2 AB and XY, BC and YZ, CA and ZX are pairs of corresponding sides in two similar triangles.

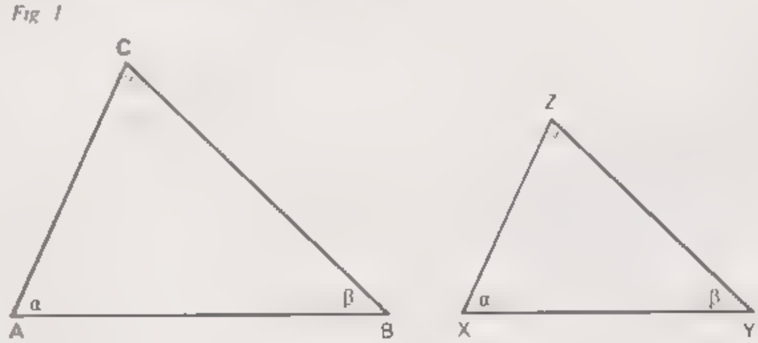


Fig. 2

2 If the sides of two similar triangles are measured, it will be found that the ratio of a pair of corresponding sides is equal to the ratio of any other pair of corresponding sides. Thus, from Fig. 2 2.E.2

$$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX}$$

It is convention to write the vertices of two similar triangles in order such that corresponding pairs of letters give corresponding sides. Thus if triangle ABC is similar to triangle XYZ, it is implied that AB and XY are corresponding sides, as are YZ and BC, and ZX and CA.

Thus, if triangles ABC and XYZ are similar

$$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX}$$

3 This property of similar triangles can be used to determine the lengths of unknown sides. For example, in Fig. 3 triangles ABE and ACD are similar. Hence 2.E.3

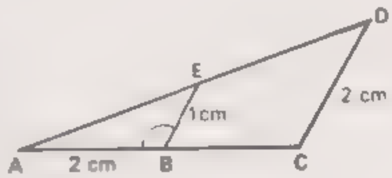


Fig 3

$$\frac{AB}{AC} = \frac{BE}{CD} = \frac{EA}{DA}$$

From the first pair of equal ratios

$$\frac{2\text{ cm}}{AC} = \frac{1\text{ cm}}{2\text{ cm}}$$

i.e. $AC = 4$ cm

4 *Criteria for Similarity* 2.E.4

i) Since the sum of the interior angles of a triangle is 180°, or π rad, (see II.A.1), if a triangle has two angles which are equal to two angles in another triangle, then the two triangles are similar.

Thus in Fig. 4, triangles ABC and ADE are similar. (Note the order of the vertices.)

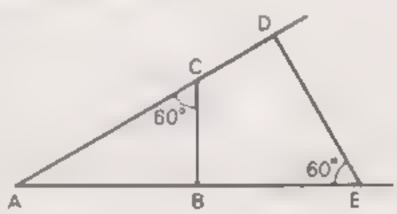


Fig 4

ii) We have seen that if two triangles are similar then the ratios of corresponding sides are equal.

The converse of this is also true, i.e. if ABC and DEF are two triangles and

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$$

then the triangles are similar. From this, we can deduce that The two triangles in Fig. 5 are similar.

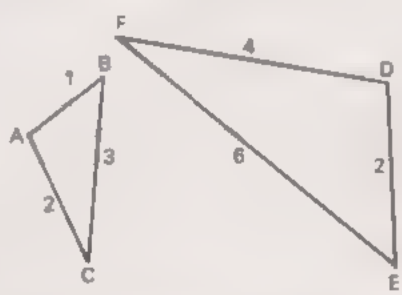


Fig. 5

5 If triangles XYZ and ABC are similar and corresponding sides are equal, i.e. 2.E.5

$$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX} = 1$$

then the triangles are said to be congruent. If two triangles are congruent, one could be picked up, turned over, if necessary, and be placed exactly on top of the other.

Thus, two triangles which are mirror images of each other are congruent.

In Fig. 6

$AB = XY$

$BC = YZ$

$CA = ZX$

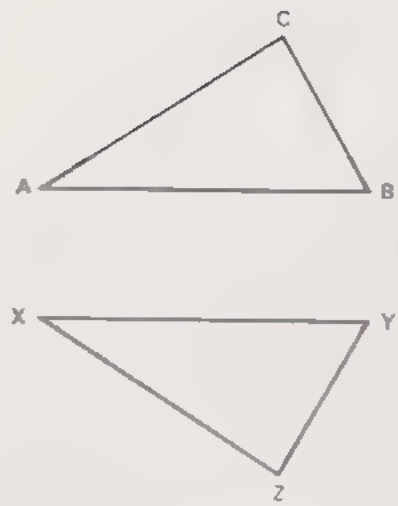


Fig. 6

6 *Criteria for congruency* 2.F.6

From certain sets of limited knowledge about the lengths of sides and the sizes of angles of two triangles, it is possible to deduce congruency.

(i) If the lengths of the sides of one triangle are equal to the lengths of another triangle, then the two triangles are congruent. Thus, the two triangles in Fig. 7 are congruent.

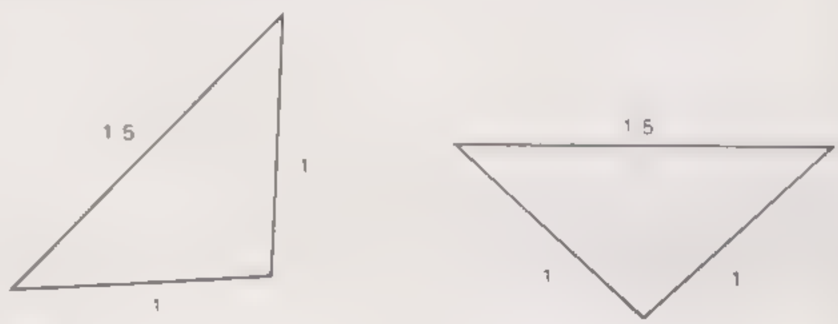


Fig. 7

(ii) If two triangles are such that two angles of one are the same as two angles of the other (i.e. they are similar triangles) *and* the lengths of a pair of corresponding sides are equal, then the triangles are congruent. Thus, the two triangles in Fig. 8 are congruent, as are triangles ABC and ADC in Fig. 9.

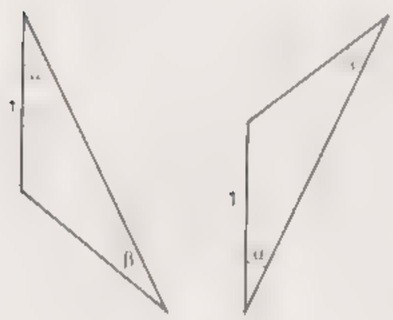


Fig. 8

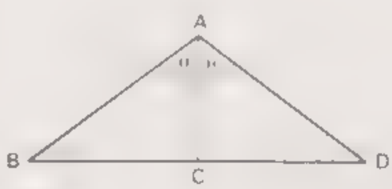


Fig. 9

(iii) If the lengths of two sides of a triangle are the same as two sides of another triangle (i.e. they have two sides of common length) *and* if one angle of one triangle is equal to one angle of the other (i.e. they have an angle in common), then the two triangles are congruent, *provided the common angle is not opposite the shorter of the two common sides.*

Thus, the two triangles in Fig. 10 are congruent, as are those in Fig. 11. (The hypotenuse is necessarily the longer side.)

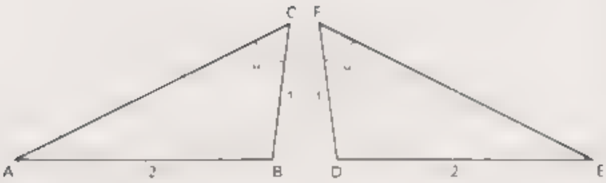


Fig 10

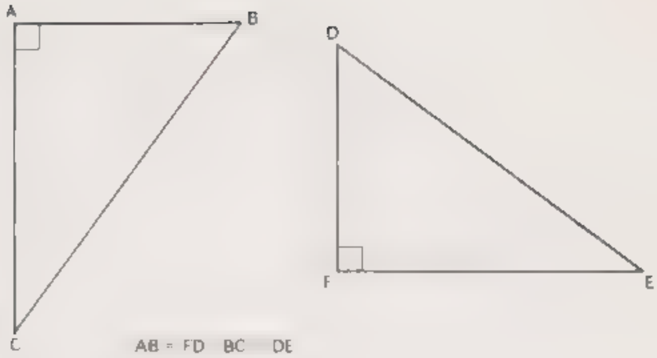


Fig 11

That congruency cannot be deduced when the common angle is opposite the shorter side is shown in Fig. 12, in which two triangles are presented having a common angle of 40° and two common sides of 2 cm and 3 cm, the 40° angle being opposite the 2 cm side.

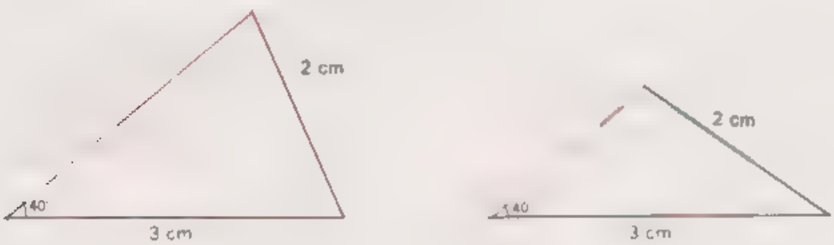
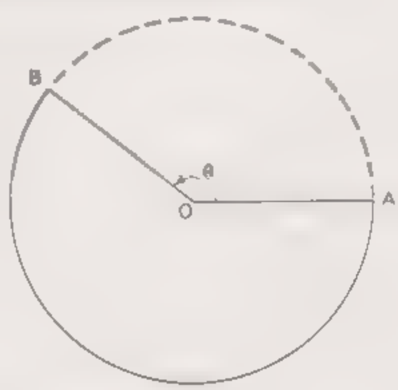


Fig 12

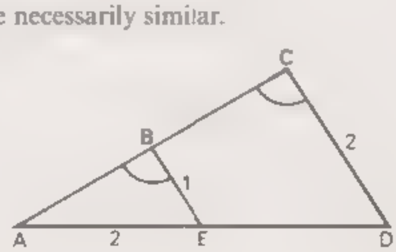
It is, of course, possible that two triangles admitting such limited knowledge are congruent; hence when confronted with such a case, all we can say is that "congruency cannot be deduced" not that "the triangles are not congruent".

Exercise Ex.2.DE

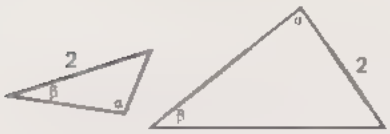
1. Which is the larger angle? 3 radians or 180°.
2. i) Convert 18° to radians. (Express your answer in terms of π.)
   ii) Convert 10 radians to degrees, correct to the nearest degree.
3. If OA, the radius of the circle in the figure is 2 units, and the length of the broken line arc AB is 3 units, what is the size of the angle θ in radians?



4. i) A pair of similar triangles are necessarily congruent. (True False)
   ii) A pair of congruent triangles are necessarily similar. (True False)
5. By using the property of similar triangles, find the length of ED.

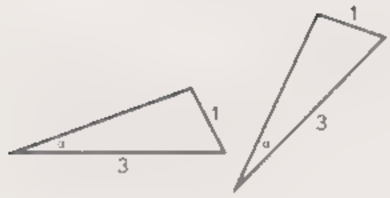


6. For which of the following pairs of triangles can congruency be deduced?



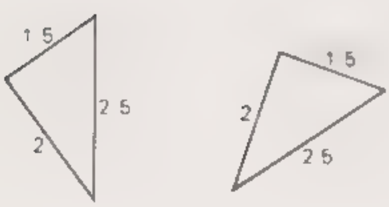
Pair A

Pair B



Pair C

Pair D

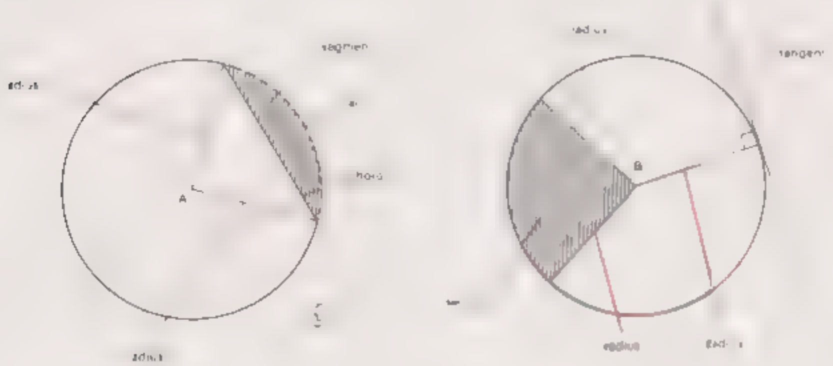


Pair E

Answers to Exercises for Section 2

Ex II ABC

1. 6 in^2
2. i) Both figures are parallelograms between a pair of parallel lines and have the same base.
   ii) 20 in^2
3. i) 28.3 in^2 ii) 9.4 in
4. It is *nine* times as great.
5.



Ex II DE

1. 180°
2. i) $18° = \frac{\pi}{10}$ rad ii) 10 rad = 573°, to the nearest degree.
3. 1.5 rad
4. i) FALSE ii) TRUE
5. $ED = 2$
6. A: No [Corresponding side not specified: criterion ii]
   B: Yes [Shorter side not opposite angle: criterion iii]
   C: No [Shorter side opposite angle: criterion iii]
   D: Yes [Longer side opposite angle: criterion iii]
   E. Yes [Three sides equal: criterion i]

Section 3 Graphs

A Cartesian Axes

Introduction

This section explains how to represent a point in a plane by a pair of Cartesian co-ordinates (x, y). Intro. 3.A

1 We can describe the location of any point in a plane (a flat 2-dimensional region, like the top of a table) by specifying the perpendicular distance from each of two intersecting perpendicular straight lines. We conceive of these straight lines as being horizontal and vertical. (Fig. 1.) 3.A.1

The horizontal line is called the x-axis and the vertical line is called the y-axis. The point of intersection of the two axes is called the origin and is marked with an O.

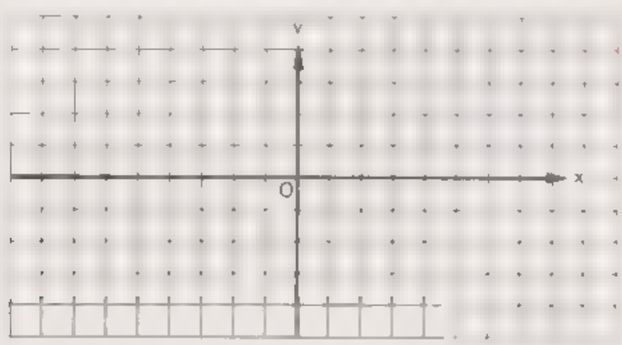


Fig. 1

Perpendicular distances of points to the right of the y-axis are designated positive; distances to the left are designated negative. Perpendicular distances of points above the x-axis are designated positive; distances to points below are designated negative.

The x-axis and y-axis are known as Rectangular Cartesian Axes after René Descartes (1596–1650), a French mathematician and philosopher.

2 The perpendicular distances of a point from the axes are written in parentheses, the distance from the y-axis being written first. From Fig. 2, we can specify 3.A.2

P as $(5, 3)$
Q as $(-4, 2)$
R as $(-7, -3)$
T as $(3, -3)$

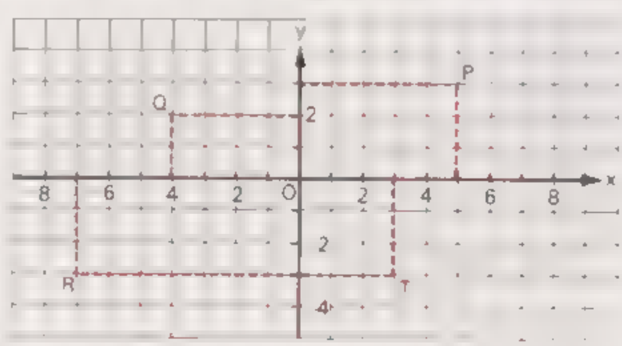


Fig. 2 Cartesian Co-ordinates

$(5, 3)$ are the Cartesian Co-ordinates of P. Note that $(5, 3)$ is different from $(3, 5)$.

To each point in the plane there corresponds a *unique* pair of co-ordinates, and to each pair of coordinates, there corresponds just one point in the plane. For this reason we often talk about the point $(2, 1)$, for example, without any ambiguity.

The first co-ordinate, since it measures distance from the y-axis i.e. along the x-axis, is called the x co-ordinate. The second co-ordinate is called the y co-ordinate. We refer to the co-ordinates of *any* point in the plane by (x, y). Note that in Fig. 2 we have the same scale on each axis; this is not always the case as you will see in later examples.

B The Graph of an Equation

Introduction Intro. 3.B

This section explains what is meant by the graph of an equation; the term "slope" is defined and the graphs of equations like

$y = ax + b$ and $y = ax^2 + bx + c$

are discussed.

1 From equations like 3.B.1

$y = 2x - 3$

$y = 3x^2 + 2x - 1$

We can obtain a y-value for any x-value we like, rather like an input–output situation. (Fig. 3.)

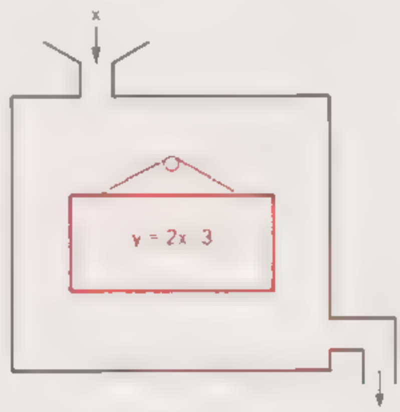


Fig 3 Input Output

Example 1

Here is a short table of values from $y = 2x - 3$

x	y	
-2	-7	$\rightarrow (-2, -7)$
-1	-5	$\rightarrow (-1, -5)$
0	-3	$\rightarrow (0, -3)$
1	-1	$\rightarrow (1, -1)$
2	1	$\rightarrow (2, 1)$

For each xy- pair we can specify a pair of coordinates.

These co-ordinates are plotted in Fig. 4. They all lie on a straight line, the graph of $y = 2x - 3$.

In general, the graph of any equation of the form $y = ax + b$ is a straight line. It is sufficient to plot *any* two points to draw the line.

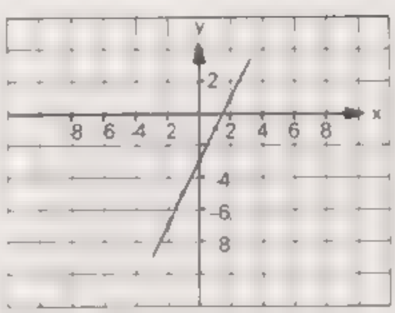


Fig 4

If (x_1, y_1) and (x_2, y_2) are the co-ordinates of *any* two points on a straight line, then the ratio

$$\frac{y_1 - y_2}{x_1 - x_2}$$

is called the *slope* of the straight line to the x-axis.

Fig. 5 shows a straight line whose equation is

$y = ax + b$

and which passes through the points A and B whose co-ordinates are (x_1, y_1) and (x_2, y_2), respectively.

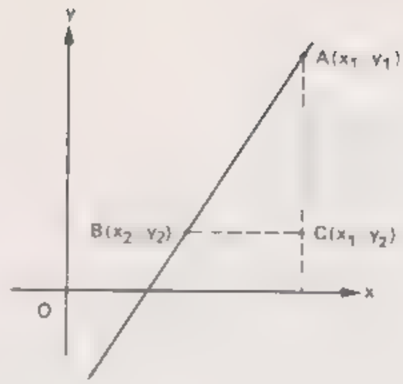


Fig 5

What is the slope of such a line?
Since A and B lie on the line

$$y_1 = ax_1 + b \qquad (1)$$

and

$$y_2 = ax_2 + b \qquad (2)$$

Therefore, by subtracting (2) from (1)

$$y_1 - y_2 = (ax_1 + b) - (ax_2 + b)$$
$$= a(x_1 - x_2)$$

Hence,

$$\text{slope} = \frac{y_1 - y_2}{x_1 - x_2} = a \qquad (3)$$

i.e. the slope of the graph of $y = ax + b$ is equal to a.

Example

The slope of $y = -2x + 3$ is -2. To check this we need to specify two points on the line (Fig. 6).

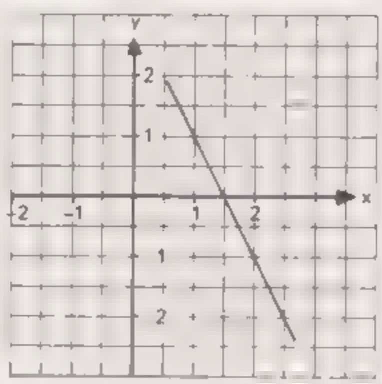


Fig. 6 $y = -2x + 3$

If

$x_1 = 1$, $y_1 = -2 \times 1 + 3 = 1$

If

$x_2 = 2$, $y_2 = -2 \times 2 + 3 = -1$

Hence

(x_1, y_1) is $(1, 1)$

and

(x_2, y_2) is $(2, -1)$

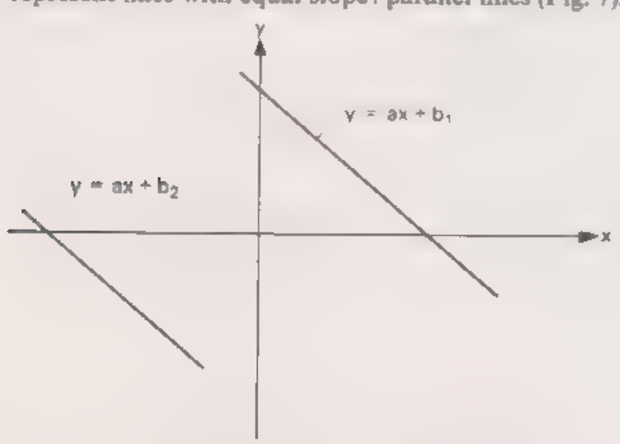
and

$$\text{slope} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{1 - (-1)}{1 - 2} = \frac{2}{-1} = -2$$

From (3), since the slope of $y = ax + b$ is independent of b

$y = ax + b_1$ and $y = ax + b_2$

represent lines with equal slope: parallel lines (Fig. 7).



See also 4.A.1 where slope is mentioned again.

2 *Example 2*

The graph of $y = \frac{1}{2}x^2 + 1$ can also be obtained by plotting a few points, if we *assume* that the points may be joined by a smooth curve.

x	y
-3	$5\frac{1}{2}$
-2	3
-1	$1\frac{1}{2}$
0	1
1	$1\frac{1}{2}$
2	3
3	$5\frac{1}{2}$

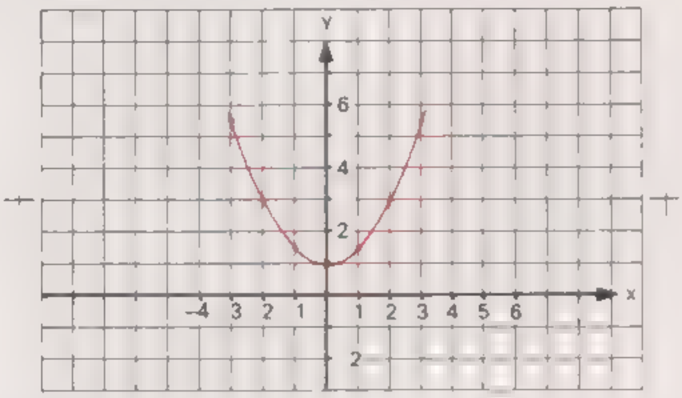


Fig. 8

In general, the graph of any equation of the form

$y = ax^2 + bx + c$

where a is not zero, has a similar shape to that in Fig. 8; the graphs of such equations are called parabolas.

C Representing Data by a Graph

Introduction Intro. 3.C

In this section we consider how to represent the data gained from simple experiments.

We meet "direct proportionality" and "slope" again.

1 In an experiment to investigate the relationship between two physical quantities one often observes the values of one quantity for specified values of the other. The table below gives corresponding values for the distance, d, travelled in time, t, by a body moving with zero acceleration (i.e. at constant speed). From this table 6 pairs of co-ordinates are found.

t (in seconds)	d (in metres)	
0	0	$\longrightarrow (0, 0)$
1	3.6	$\longrightarrow (1, 3.6)$
2	8.4	$\longrightarrow (2, 8.4)$
3	12.4	$\longrightarrow (3, 12.4)$
4	15.8	$\longrightarrow (4, 15.8)$
5	20.2	$\longrightarrow (5, 20.2)$

These co-ordinates can be plotted. In order to do this we must choose suitable scales for each axis. If, for example, one wishes to read off the value of t when d is 17 metres, the scale must be such that 17 m is easily read; i.e. lies on a subdivision of the graph grid. Another general consideration is that one should use a good proportion of the available graph paper in order to make plotting and reading points as accurate as possible. The graph in Fig. 9 shows how the data above can be plotted in such a way as to demonstrate the two points above. Note that scales are prominently displayed and axes are clearly labelled.

2 By inspecting the graph we can see that the distance travelled is *directly proportional* to the time passed, i.e.

$d \propto t$

$d = vt$

where v is the constant of proportionality, the constant speed of the body. A straight line is drawn through the points; its slope, gives the constant velocity, v. (N.B. The straight line can be drawn by using a transparent ruler and trying to get as many points above the line as below, so that the sum of the distances of the points from the line is as small as possible.) The slope is calculated by constructing a right-angled triangle at points where the straight line passes through conveniently read subdivisions, in this case v is

$$\frac{(17.6 - 2.2)\,\text{m}}{(4.4 - 0.5)\,\text{s}} = \frac{15.4}{3.9}\,\text{ms}^{-1} = 3.95\,\text{ms}^{-1}$$

Note that it is a convention to represent the observed values on the vertical axis and the pre-selected values on the horizontal axis.

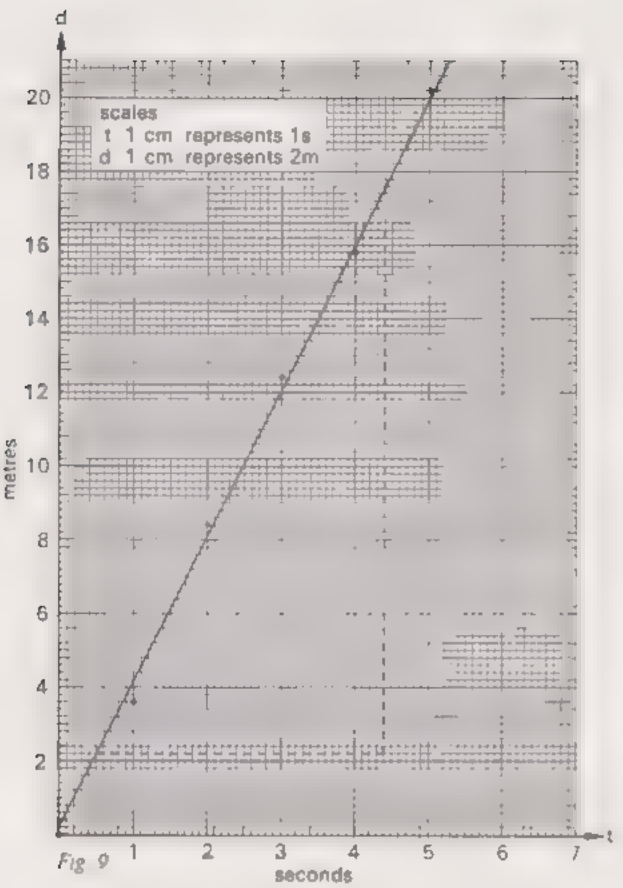


Fig 9

Exercise

1 Classify the following equations as "straight line", "parabola", or "neither" by considering their graphs.

i) $y = 3x - 5$ ii) $y = 3x^2 - 2x + 3$

iii) $y = (2x + 6)^2$ iv) $y = 3$

v) $y = (x - 1)(x + 1)$ vi) $y = x$

2 Draw the graph of $y = -2x - 3$.

3 What is the slope of the graphs of the following equations?

i) $y = 6x + 1$ ii) $y = -10x - 2$

4 Evaluate the slopes of the straight lines which pass through the pairs of points indicated.

i) $(0, 3)$ and $(-10, 1)$

ii) $(7, 3)$ and $(-1, 4)$

5 Plot the following data on a sheet of centimetre–millimetre graph paper and deduce the value of l when $p = 7.6$.

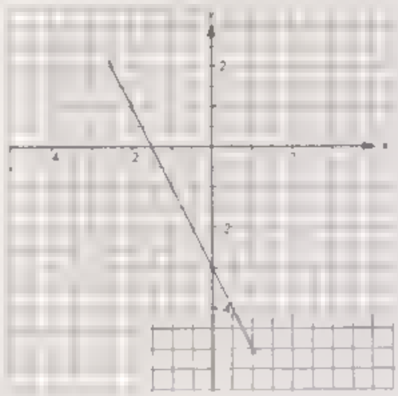
l	1	2	3	4	5	6	7	8	9
p	1.9	3.6	5.8	7.9	9.7	11.8	13.1	15.9	18.3

Answers to Exercises for Section 3

Section 3 Answers

1 i) straight line ii) parabola iii) parabola
iv) straight line v) parabola vi) straight line

2 Any two points are sufficient to specify the line.



graph of $y = 2x - 3$

3 i) 6 ii) −10

4 i) 0.2 ii) −0.125

5 Between 3.85 and 3.95.

Section 4 Trigonometry

A Tangent, Sine and Cosine

Introduction Intro. 4.A

In this section we define the tangent, sine and cosine of an angle. Tangent and slope are discussed as are the approximations

$\tan \theta \approx \theta$

$\sin \theta \approx \theta$

where

θ is measured in radians. The graphs of tangent, sine and cosine are presented and the term "periodic" is defined.

1 In Fig. 1, triangle OPQ is right-angled at Q and P has co-ordinates (x, y). Hence the co-ordinates of Q are (x, O). Angle POQ, θ, is less than $\frac{\pi}{2}$ rad; i.e. θ is acute 4.A.1

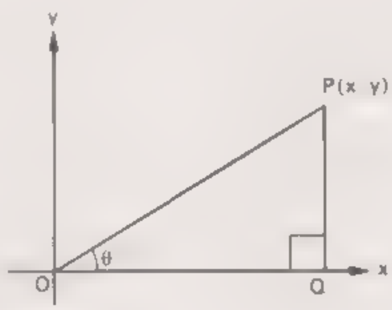


Fig. 1

OP has length r, i.e. $r = +\sqrt{x^2 + y^2}$ by Pythagoras' Theorem. {See IV.C.2.}

We define the tangent of θ, abbreviated to $\tan \theta$, by

$$\tan \theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{y}{x}$$

We define the sine of θ, abbreviated to $\sin \theta$, by

$$\sin \theta = \frac{\text{opposite side}}{\text{hypotenuse side}} = \frac{y}{r}$$

We define the cosine of θ, abbreviated to $\cos \theta$, by

$$\cos \theta = \frac{\text{adjacent side}}{\text{hypotenuse side}} = \frac{x}{r}$$

As long as θ remains constant, each ratio remains the same for any position of $P(x, y)$. Fig. 2 shows two positions of P, P_1 and P_2. Triangles OQ_1P_1 and OQ_2P_2 are similar and hence

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = \tan \theta$$

$$\frac{y_1}{r_1} = \frac{y_2}{r_2} = \sin \theta$$

$$\frac{x_1}{r_1} = \frac{x_2}{r_2} = \cos \theta$$

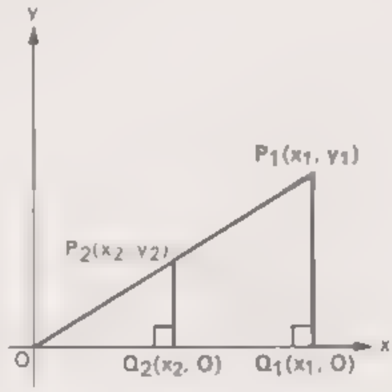


Fig. 2

In Fig. 3, P_2R has been drawn parallel to the x-axis. Triangles OQ_2P_2, OQ_1P_1 and P_2RP_1 are similar and hence

$$\frac{y_2}{x_2} = \frac{y_1}{x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

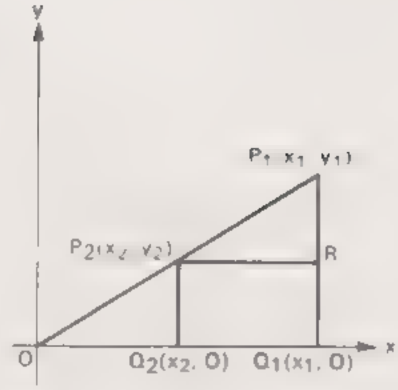


Fig 3

Hence, the slope of a straight line is the same as the tangent of the angle measured from the x-axis anticlockwise to the line.

2 By considering the triangles drawn, Figs. 4 and 5, and the definitions above, the following *useful values* can be obtained 4.A.2

Note that the convention of dropping the unit symbol has been used when an angle is measured in radians. Also note that the sum of the interior angles of a triangle is $\pi = 180°$.

θ	$\tan\theta$	$\sin\theta$	$\cos\theta$
0	0	0	$1\ (x = r)$
$\frac{\pi}{6} = 30°$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4} = 45°$	1	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
$\frac{\pi}{3} = 60°$	$\sqrt{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$

Note that $\sqrt{2}$ and $\sqrt{3}$ have been left in the denominators since it is more convenient to divide such roots by an integer than to divide an integer by such a root, when the decimal form is required

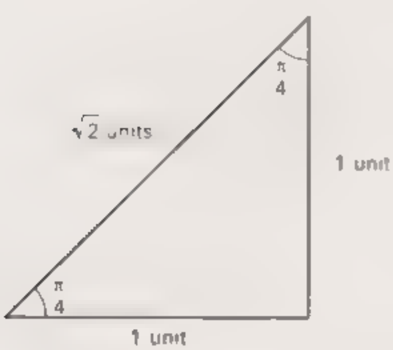


Fig 4

Fig 5

3 Books of mathematical tables list the values of tan, sin and cos for angles measured in degrees and for angles measured in radians. (II.D.2.) For *small angles* the following approximations are useful provided θ is measured in *radians* 4.A.3

$\tan\theta \approx \theta$

$\sin\theta \approx \theta$

To indicate how small θ has to be, here are some actual values.

θ (rad)	$\tan\theta$	$\sin\theta$
0.05	0.0500	0.0500
0.10	0.1003	0.0998
0.15	0.1511	0.1494
0.30	0.3093	0.2955
0.60	0.6841	0.5646

If, for example, an accuracy of 1 part in 100 is sufficient for $\tan\theta$ values, then the approximation $\tan\theta \approx \theta$ is true for θ less than 0.30 radians, about 17°.

We can see why the approximations for sin and tan hold for small angles by considering the triangle below in which θ is less than 0.2 rad. For such angles the length of arc AC, s, is approximately equal to CB, y (Fig. 6).

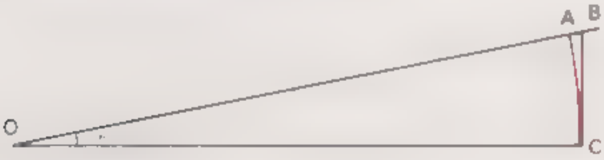


Fig. 6

4 The definitions of tan, sin and cos can be extended to angles which are not acute. If $P(x, y)$ is *any* point in the plane and θ is the angle measured from the positive x-axis to OP then

$\tan\theta = \frac{y}{x} \quad \sin\theta = \frac{y}{r} \quad \cos\theta = \frac{x}{r}$

Fig. 7 shows θ acute, whereas in Fig. 8 θ lies between π and $\frac{3\pi}{2}$

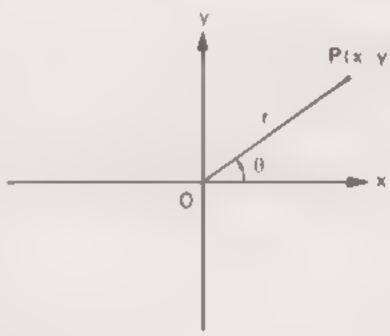


Fig. 7

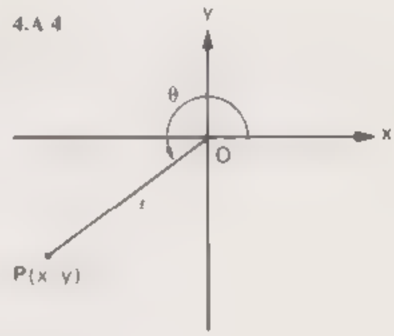


Fig. 8

Examples

For points P_2 $(-5, 12)$, P_3 $(-4, -3)$ and P_4 $(4, -4)$ in Fig. 9 write down the values of tan, sin and cos.

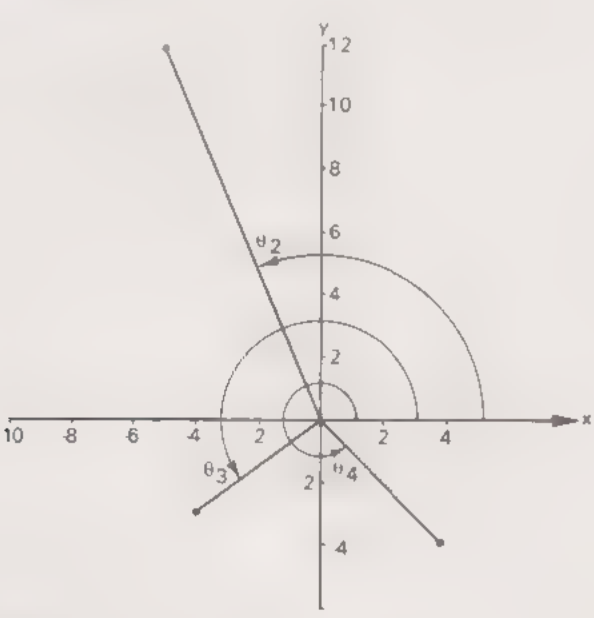


Fig. 9

Point	tan	sin	cos	Angle
$(-5, 12)$	$\frac{12}{-5} = -2.4$	$\frac{12}{13}$	$\frac{-5}{13}$	θ_2
$(-4, -3)$	$\frac{-3}{-4} = 0.75$	$\frac{-3}{5} = -0.6$	$\frac{-4}{5} = -0.8$	θ_3
$(4, -4)$	$\frac{-4}{4} = -1$	$\frac{-4}{\sqrt{32}} = \frac{-\sqrt{2}}{2}$	$\frac{4}{\sqrt{32}} = \frac{\sqrt{2}}{2}$	θ_4

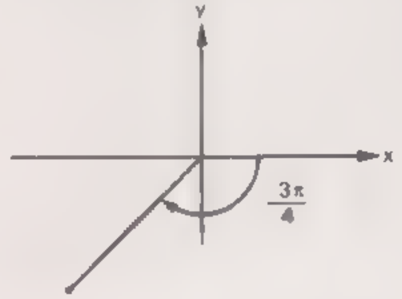
5 Note that since the definition of $\tan\theta$ involves division by x, $\tan\theta$ is *not defined* when x is zero. In this position OP lies along the y-axis and the corresponding values of θ for which $\tan\theta$ is not defined are 4.A.5

$\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$

6 Negative angles are measured clockwise from Ox; in Fig. 10, a negative angle of $\frac{-3\pi}{4}$ is shown. 4.A.6

$\tan\theta$ is not defined for

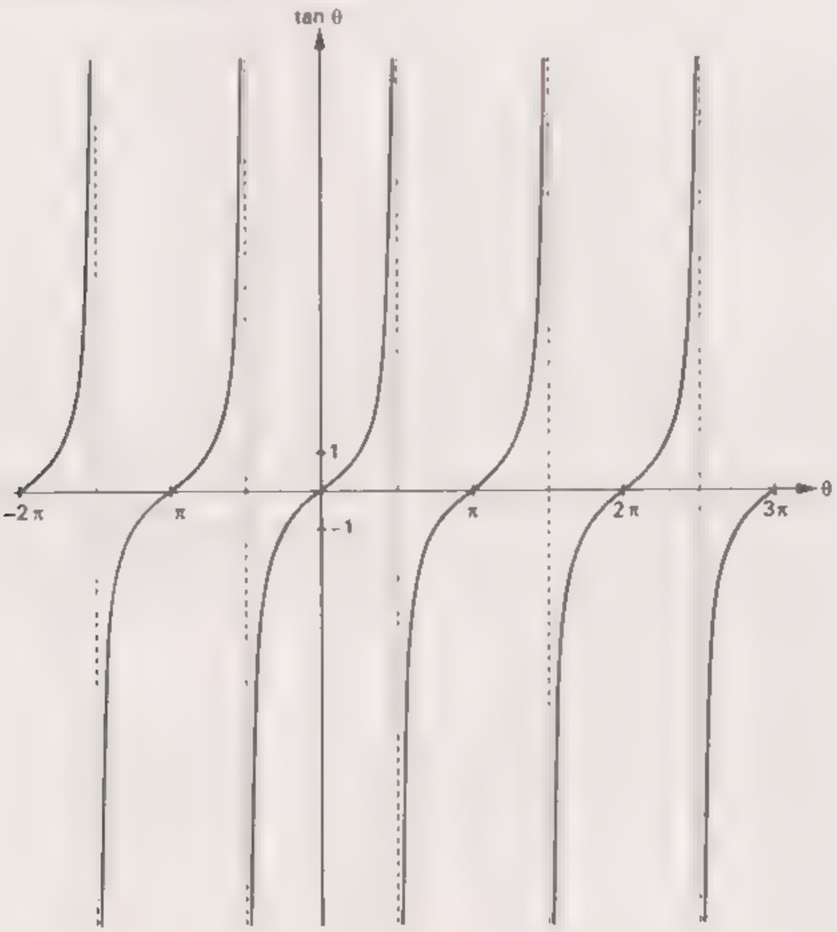
$\theta = \frac{-\pi}{2}, \frac{-3\pi}{2}, \frac{-5\pi}{2}, \frac{-7\pi}{2}, \ldots$



7 The graphs of $\tan\theta$, $\sin\theta$ and $\cos\theta$ are shown below (Figs 11, 12 and 13). Note that each repeats itself like wallpaper; we say they are periodic. 4.A.7

Since the graph of $\tan\theta$ starts to repeat itself after an interval of π along the θ-axis, we say $\tan\theta$ has period π.

The period of $\sin\theta$ and $\cos\theta$ is 2π.



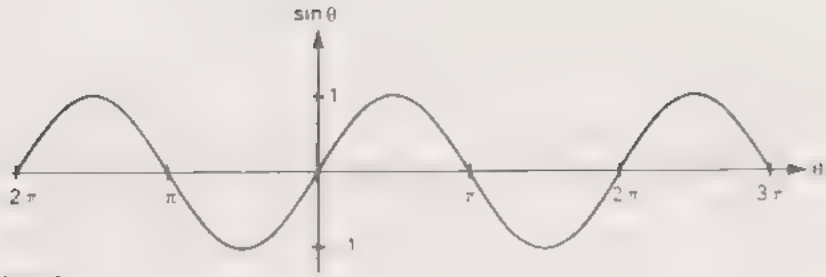


Fig 12

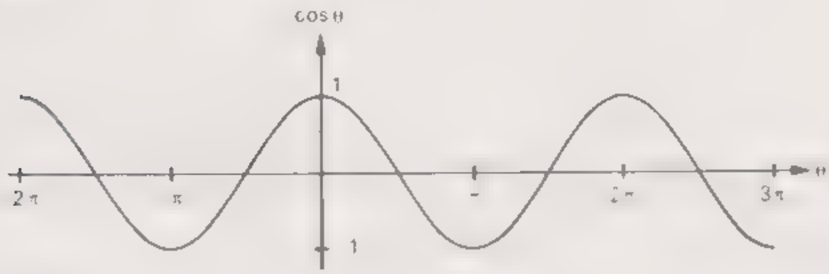


Fig 13

Note that $\sin\theta$ and $\cos\theta$ never achieve values greater than 1 (x and y are never greater than r), and that for the values of θ for which $\tan\theta$ is not defined, vertical broken lines are drawn: these lines are such that the graph of $\tan\theta$ never crosses them but it comes as close to them as we like; in a sense these lines are boundaries for the repeated shape of the graph. Such lines are called asymptotes.

From the graphs it is easy to see the signs of tan, sin and cos for the ranges of values 0 to $\frac{\pi}{2}$, $\frac{\pi}{2}$ to π, π to $\frac{3}{2}$ and $\frac{3\pi}{2}$ to 2π; they are summarized below.

sine positive; cosine, tangent } negative	All positive
tangent positive; cosine, sine } negative	cosine positive; tangent, sine } negative

Exercise Ex. 4.A

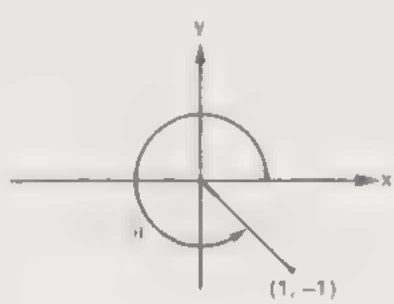
1 When are the approximations
$\tan\theta \approx \theta$
$\sin\theta \approx \theta$
valid? (Give a two-point answer.)

2 Evaluate the following by sketching the appropriate triangles.

i) $\tan\frac{\pi}{6}$ ii) $\tan 30°$

iii) $\cos\frac{\pi}{3}$ iv) $\sin 30°$

3 Write down the values of
i) $\tan\theta$
ii) $\sin\theta$
iii) $\cos\theta$
for the angle shown.



4 Sketch the graphs of $\sin\theta$ and $\tan\theta$ for θ in the range 0 to $\frac{5\pi}{2}$.

5 What is the *sign* of

i) $\cos 278°$
ii) $\sin 1$ rad
iii) $\tan 300°$
iv) $\cos\frac{5\pi}{6}$?

B Trigonometrical Tables

Introduction Intro. B.B

In this section we explain how to read tangent, since and cosine tables.

1 Pages 6, 7, 8, 9, 10 and 11 of *Clark's Tables* present the values of the sine, cosine and tangent of angles between 0° and 90°. 4.B.1

These tables are read in much the same way as the logarithm and antilogarithm tables. (See 1.B.5.) On pages 6 and 7 is the sine table; you can see that degrees are printed in the far left column and that heading the columns of the main body of the table are decimal subdivisions of the degree. The subdivisions are also written in the unit of angle the minute — in 1 degree there are 60 minutes. Note that the abbreviation for minute is ′; thus 60′ = 1°.

2 *Example* 4.B.2

What is the value of $\sin 52° 46'$?

Locate the 52° row and the column above the main body which is headed by the nearest number of minutes below 46′ (in this case 42′): $\sin 52° 42' =$ 0.795 5. Since $\sin\theta = \frac{y}{r}$ and, as an acute angle θ increases, y increases, we need to know the extra bit to *add* to 0.795 5 to obtain the value of $\sin 52° 46'$.

In the mean differences locate the 4′ column (46 − 42 = 4) and add the reading in the 52° row to 0.795 5; i.e.

$$\sin 52° 46' = 0.795\,5 + 0.000\,7$$
$$= 0.796\,2.$$

	0′ 0.0°	6′ 0.1°	12′ 0.2°	18′ 0.3°	24′ 0.4°	30′ 0.5°	36′ 0.6°	42′ 0.7°	48′ 0.8°	54′ 0.9°	Mean Differences 1′	2′	3′	4′	5′
45°	0.7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2	4	6	8	10
46	0.7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2	4	6	8	10
47	0.7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2	4	6	8	10
48	0.7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2	4	6	8	10
49	0.7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	2	4	6	8	9
50°	0.7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	2	4	6	7	9
51	0.7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2	4	5	7	9
52	0.7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2	4	5	7	9
53	0.7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2	3	5	7	9
54	0.8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2	3	5	7	8
55	0.8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	2	3	5	7	8

Although values obtained in the main body of the table for 6′ subdivisions of the degree are accurate to four decimal places, values obtained using the mean difference column may be in error by a unit or two in the fourth decimal place.

3 The cosine table on pages 8 and 9 is read in just the same way as the sine table except that the mean differences must be *subtracted* since $\cos\theta = \frac{x}{r}$ and as an acute angle θ increases, x *decreases*. 4.B.3

Example

What is the value of cos 27° 29′?

	0′ 0.0°	6′ 0.1°	12′ 0.2°	18′ 0.3°	(24′) 0.4°	30′ 0.5°	36′ 0.6°	42′ 0.7°	48′ 0.8°	54′ 0.9°	SUBTRACT Mean Differences 1′	2′	3′	4′	5′
26	0.8988	8980	8973	8965	8957	8949	8942	8943	8926	8918	1	3	4	5	6
27	0.8910	8902	8894	8886	8878	8870	8862	8854	8846	8838	1	3	4	5	7
28	0.8829	8821	8813	8805	8796	8788	8780	8771	8763	8755	1	3	4	6	7
29	0.8746	8738	8729	8721	8712	8704	8695	8686	8678	8669	1	3	4	6	7
30°	0.8660	8652	8643	8634	8625	8616	8607	8599	8590	8581	1	3	4	6	7

cos 27° 24′ = 0.887 8

Correction from 5′ column of mean differences in 27° row is 0.000 7; so

cos 27° 29′ = 0.887 8 − 0.000 7

= 0.887 1

4 The tangent table on pages 10 and 11 is read in much the same way as the sine table, mean differences are *added* since $\tan\theta = \frac{y}{x}$ and as an acute angle θ, y increases and x decreases. 4.B.4

For angles less than 45°, the table is read in just the same way as the sine table. You should be able to check that

tan 37° 39′ = 0.770 1 + 0.001 4

= 0.771 5

For angles greater than 45° the tangent exceeds 1. To save space the whole number only appears in the first column of the main body of the table. On occasions the mean difference *may* cause the whole number to change

Example

What is the value of tan 63° 27′?

tan 63° 24′ = 1.997 0

tan 63° = 1.997 0 + 0.004 4

= 2.001 4

	0′ 0.0°	6′ 0.1°	12′ 0.2°	18′ 0.3°	(24′) 0.4°	30′ 0.5°	36′ 0.6°	42′ 0.7°	48′ 0.8°	54′ 0.9°	Mean Differences 1′	2′	(3′)	4′	5′
45	1.0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
61	1.8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13	26	38	51	64
62	1.8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	14	27	41	55	68
63	1.9626	9711	9797	9883	9970	0057	0145	0233	0323	0413	15	29	44	58	73
64	2.0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	16	31	47	63	78
65	2.1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17	34	51	68	85

Note that when the suppressed whole number part of an entry changes along a row, a bar is placed over the digit in the tenths column of the entry.

Notice that for tangents greater than 75° mean differences are not printed. As you can see from the values in the main body of the table the tangent in this region is increasing rapidly. The method of mean differences in this region gives values which are much more inaccurate than the 1 or 2 units in the fourth decimal place. Hence, to find a value such as tan 81° 39′, a more sophisticated set of tables is used.

Exercise Ex. 4.B

Use the appropriate tables to evaluate

i) sin 29° 32′ ii) sin 49° 51′
iii) cos 81° 19′ iv) cos 17° 13′
v) tan 71° 34′ vi) tan 18° 13′

C Identities 4.C.1

Introduction

In this section the following identities are discussed

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

1 An identity is an equation whose solution is all values of the variable for which both sides of the equation are defined.

Since

$$\tan\theta = \frac{y}{x} \quad (1)$$

$$\sin\theta = \frac{y}{r} \quad (2)$$

and

$$\cos\theta = \frac{x}{r} \quad (3)$$

we can deduce that

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad (4)$$

for all values of θ, except $\theta = +\frac{\pi}{2}, -\frac{\pi}{2}, +\frac{3\pi}{2}, -\frac{3\pi}{2}, \ldots$ for which both sides are not defined. (4) is an example of an identity.

2 Pythagoras' Theorem states that in a right-angled triangle the area of the square on the hypotenuse equals the sum of the areas of the squares on the other two sides (Fig. 1). 4.C.2

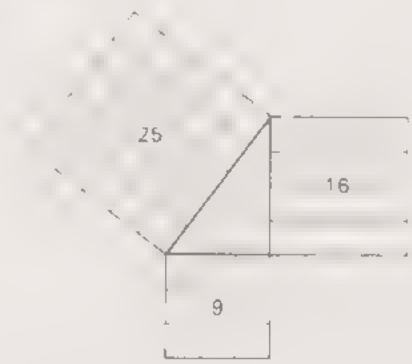


Fig. 1

Algebraically, from Fig. 2

$x^2 + y^2 = r^2$ (5)

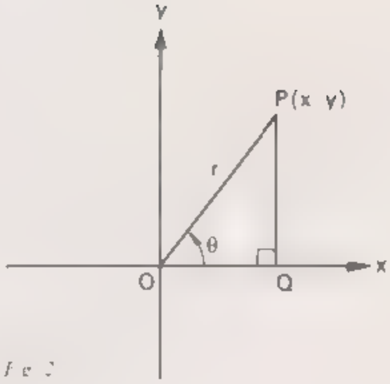


Fig. 2

3 Divide both sides (5) by r^2 4.C.3

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2}$$

i.e.

$\cos^2\theta + \sin^2\theta = 1$ (6)

(6) is an identity for acute angles

Although we started from Pythagoras' Theorem applied to triangle OPQ, (6), in fact, is an identity for *all* angles.

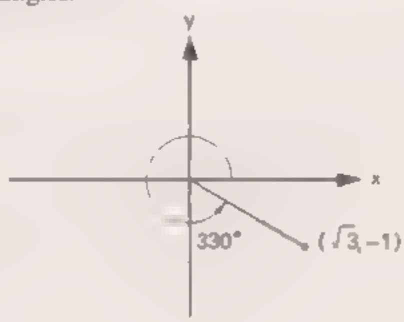
For example, from Fig. 3, for 330°

$$\cos 330° = \frac{x}{r}$$

$$= \frac{\sqrt{3}}{+\sqrt{(\sqrt{3})^2 + (-1)^2}} = \frac{\sqrt{3}}{2}$$

$$\sin 330° = \frac{y}{r}$$

$$= \frac{-1}{+\sqrt{(\sqrt{3})^2 + (-1)^2}} = -\frac{1}{2}$$



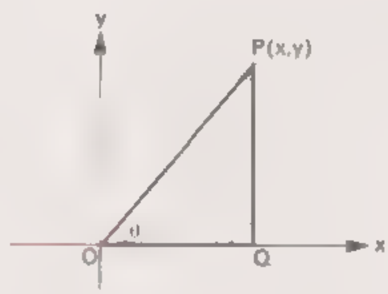
So

$$\cos^2 330° + \sin^2 330°$$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 = 1$$

4 If $\alpha + \beta = \frac{\pi}{2}$ 4.C.4

α and β are called complementary angles. In triangle OQP, $\angle QPO = \frac{\pi}{2} - \theta$. In terms of opposite, adjacent and hypotenuse sides we have



$$\sin\theta = \frac{PQ}{OP}$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \frac{PQ}{OP}$$

$$\cos\theta = \frac{QO}{OP}$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \frac{QO}{OP}$$

Hence $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ (7)

and $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$ (8)

(7) and (8) are identities for all angles. For example, from Fig. 5

$\sin 30° = \frac{1}{2}$

and

$\cos 60° = \cos(90° - 30°) = \frac{1}{2}$

Similarly

$\cos 30° = \frac{\sqrt{3}}{2}$

and

$\sin 30° = \cos(90° - 30°) = \frac{\sqrt{3}}{2}$

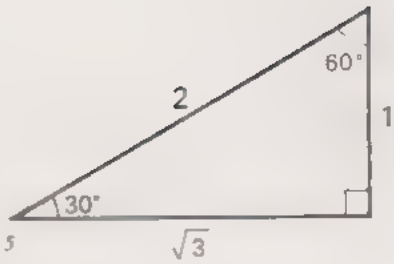


Fig 5

5 *Examples* 4.C.5

i) If $\sin\theta = \frac{5}{13}$ and θ is acute, what is the value of $\cos\theta$?

Since

$\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

and

$\cos\theta = +\sqrt{1 - \sin^2\theta}$

since θ is acute.

So

$$\cos\theta = +\sqrt{1 - \tfrac{25}{169}}$$
$$= +\sqrt{\tfrac{144}{169}}$$
$$= \tfrac{12}{13}$$

ii) Given that $\cos 74° = 0.275\,6$, what is the value of $\sin 16°$?

Since

$16° + 74° = 90°$

$\cos(90° - \theta) = \sin\theta$

$\cos 74° = \sin 16° = 0.275\,6.$

Exercise Ex. 4.C

Use the identities of this section to answer the following.

1 If $\sin\theta = 0.8$ and $\cos\theta = 0.6$, what is the value of $\tan\theta$?

2 If $\sin\theta = 0.50$ and θ is acute, what is the value of $\cos\theta$ to two decimal places?

3 $\cos 81° = 0.156\,4$
$\sin 9° =$ ______

4 If $\cos\theta = -0.6$ and θ lies between 90° and 180°, what is the value of $\sin\theta$?

D Vector Quantities

Introduction Intro. 4.D

In this section we define a vector quantity by way of directed line segments and we see how to "add" vector quantities, by the "head-to-tail" rule. Vectors are then expressed in terms of their Cartesian components and we deduce that

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

where α, β and γ are the direction cosines of a vector.

1 The speed at which a car is travelling may be said to be 60 mph, but this information is not very useful since it does not tell us where the car will be in, say, five minutes time: we need to specify direction as well. 4.D.1

The combination of a speed and a direction is called *velocity*. Speed can be thought of as the magnitude of velocity.

Another physical quantity which has both magnitude and direction is *force*: a force can be pushing or pulling a body to which it is applied.

2 Physical quantities like force and velocity can be represented by directed straight line segments: the length of which represents the magnitude of the physical quantity and whose direction relative to some fixed direction represents the direction of the physical quantity. 4.D.2

For example, a velocity of $15\ m\ s^{-1}$ (see * below) in a due North East direction could be represented by the directed line segment shown in Fig. 1.

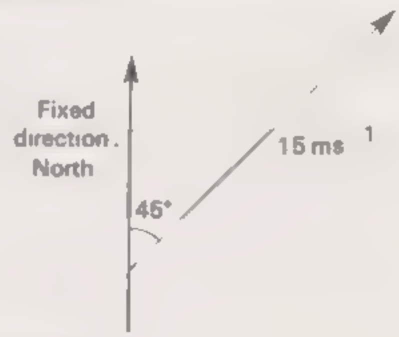


Fig. 1

For two forces acting at a point on a body, the direction of one of the forces can be used as the fixed direction.

Figure 2 shows a force of 10 N (see * below) and one of 15 N inclined at 30°, both acting away from the body.

Fig. 2

3 Directed line segments which lie in the same plane (we use the plane of the printed page!) can be "added" according to the following rule, called the head-to-tail rule. 4.D.3

To add a line segment of length b denoted by **b** to a line segment of length a denoted by **a**, transfer **b** parallel to itself so that its tail and the head of **a** meet at a point: this is shown in Fig. 3. The addition of line segment **b** to line segment **a** is the line segment shown in red, note that its head meets that of **b** and its tail coincides with that of **a**. If we denote the red line segment by **c** we have

where + denotes addition of line segments.

* *For those of you who have forgotten* m *is the abbreviation for metre,* s *is the abbreviation for second and* N *is the abbreviation for newton.*

The metre, second and newton are the units of length, time and force, respectively, in the SI system. (See the section 3 in The Handling of Experimental Data.)

As we have seen in the addition rule above, any two line segments of the same length and direction are taken to be equivalent.

Fig. 4

4 Fig. 5 shows that 4.D.4

a + **b** = **b** + **a**

since the two red line segments are equivalent.

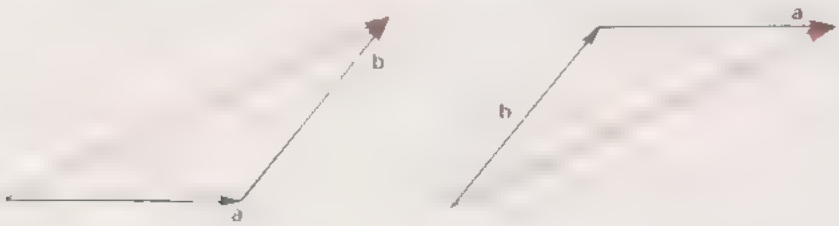


Fig. 5

5 It can be shown experimentally that the resultant of two forces represented by **a** and **b** acting at a point (i.e. the one force which has the same effect as the two) is represented by **a** + **b**. 4.D.5

We define a vector quantity as a physical quantity which is specified by a magnitude and a direction and which can be added by the "head-to-tail" rule.

6 Any number of such vector quantities can be added by this method. Fig. 6 shows the extension of the "head-to-tail" rule to 3 vector quantities; the intermediary sum **a** + **b** is shown by a broken line. (Note that from now on we denote vectors as well as line segments by **a**, **b**, etc.) 4.D.6

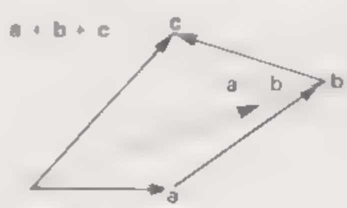


Fig. 6

Following normal practice, we have used a bold-face type to denote a vector. As you cannot reproduce bold-face lettering in your notebook, we suggest that you denote the vector **a** by a̲.

7 **The Components of a Vector** 4.D.7

In Fig. 7, Cartesian axes are set up at the tail of the vector quantity **a**.

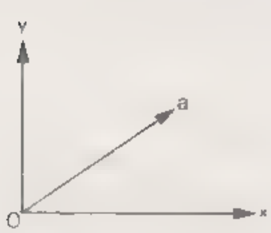


Fig. 7

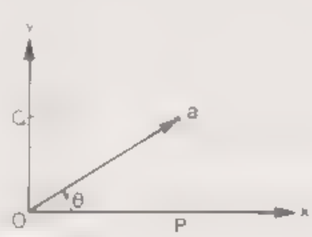


Fig. 8

If a is the length of **a** and θ is the angle of inclination of **a** to the positive x-axis (Fig. 8).

OP has length $\quad a\cos\theta$

OQ has length $\quad a\cos(90° - \theta) = a\sin\theta$.

$a\cos\theta$ is the projection of length a on the x-axis; $a\sin\theta$ is the projection of length a on the y-axis.

If $\mathbf{a}_x$ is a vector quantity of magnitude OP and direction along the positive x-axis and $\mathbf{a}_y$ is a vector quantity of magnitude OQ and direction along the positive y-axis

(Fig. 9), then

$\mathbf{a} = \mathbf{a}_x + \mathbf{a}_y$

$\mathbf{a}_x$ and $\mathbf{a}_y$ are called the component vectors of $\mathbf{a}$ in the x and y directions.

The equation

$\mathbf{a} = \mathbf{a}_x + \mathbf{a}_y$

is sometimes abbreviated to

$\mathbf{a} = [a_x, a_y]$

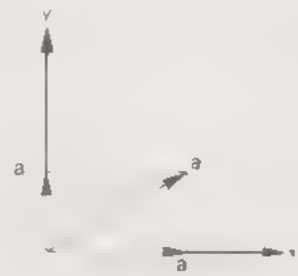


Fig. 9

So, for example, vector $\mathbf{a}$ in Fig. 10 can be specified by

$\mathbf{a} = [4, 3]$. Note that

$\mathbf{b} = [3, 4]$

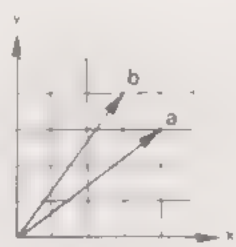


Fig. 10

[4, 3] are the components of $\mathbf{a}$. Square brackets are used to distinguish a vector from a point in this text. Note that, by Pythagoras' Theorem

$a^2 = a_x^2 + a_y^2$

and that if $\mathbf{a}$ is such that $a = 1$ (Fig. 11)

$1^2 = a_x^2 + a_y^2$

or

$1 = \cos^2\alpha + \sin^2\alpha$

i.e.

$1 = \cos^2\alpha + \cos^2\beta$

Fig. 11

8 Vector Quantities in Three-Dimensions 4.D 8

The representation of vector quantities by line segments extends to three dimensions in a natural way.

$\mathbf{a} = [a_x, a_y, a_z]$ (Fig. 12)

and from Fig. 13

$a_x = a\cos\alpha$

$a_y = a\cos\beta$

$a_z = a\cos\gamma$

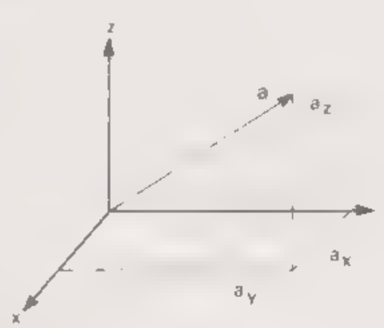


Fig. 12

where

α is the angle between **a** and Ox
β is the angle between **a** and Oy
γ is the angle between **a** and Oz

$\mathbf{a} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$

$a^2 = a_x^2 + a_y^2 + a_z^2$

and if $a = 1$

$1^2 = a_x^2 + a_y^2 + a_z^2$

or

$1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$

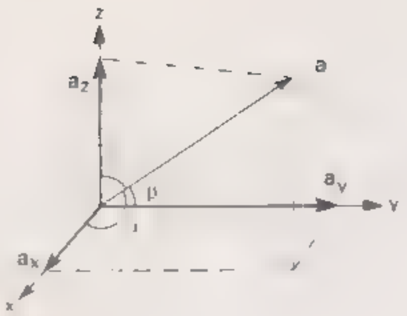


Fig 13

α, β and γ are called the direction cosines of **a**.

Example

If we know the components of **a** we can calculate the direction cosines of **a**.

If $\mathbf{a} = [4, -3, 7]$

$$\cos\alpha = \frac{a_x}{a} = \frac{4}{+\sqrt{4^2 + (-3)^2 + 7^2}} = \frac{4}{\sqrt{74}} = \frac{2\sqrt{74}}{37}$$

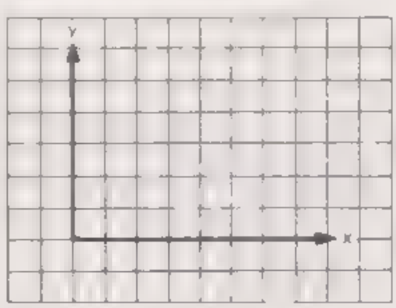
$$\cos\beta = \frac{a_y}{a} = \frac{-3}{\sqrt{74}} = \frac{-3\sqrt{74}}{74}$$

$$\cos\gamma = \frac{a_z}{a} = \frac{7}{\sqrt{74}} = \frac{7\sqrt{74}}{74}$$

Exercise Ex. 4.D

1 i) Mark in the sum **a** + **b** + **c** on the figure.

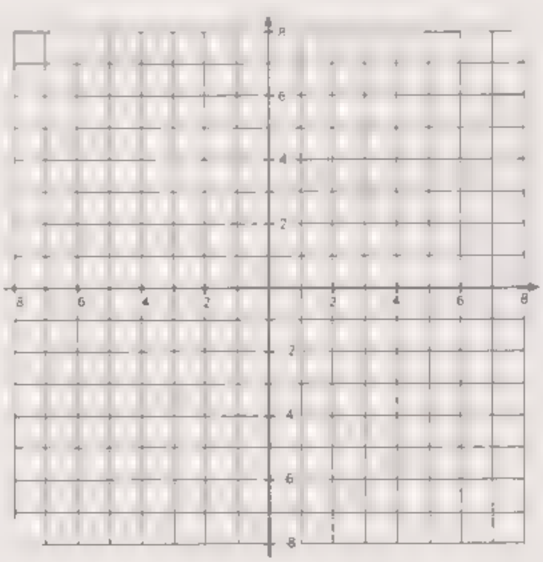
ii) Draw the vector **b** whose components are [6, 3].



2 Evaluate the direction cosines of the vector **c** whose components are $[3, -8, 2\sqrt{2}]$.

3 By drawing $\mathbf{a} = [4, 2]$ and $\mathbf{b} = [-3, 6]$ show that

$\mathbf{c} = \mathbf{a} + \mathbf{b} = [4 - 3, 2 + 6] = [1, 8]$



Answers to Exercises for Section 4

Ex. 4.A

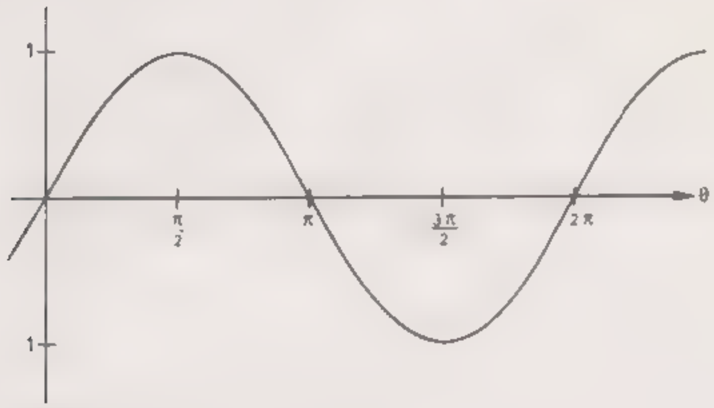
1 When θ *measured in radians is small*

2 i) $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ ii) $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

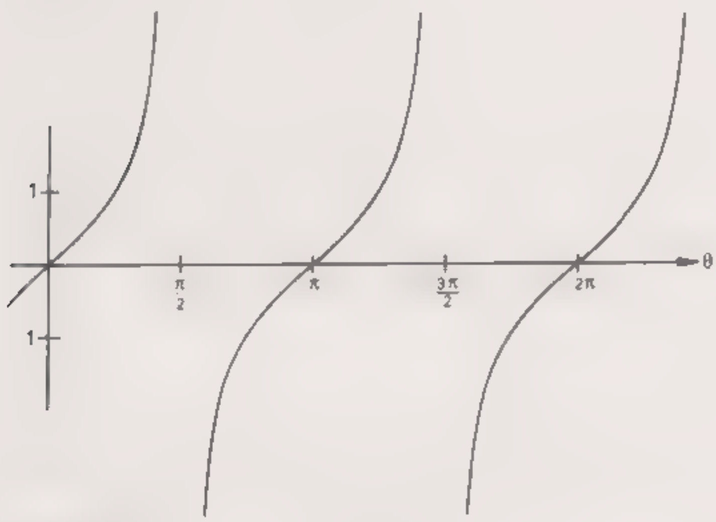
iii) $\frac{1}{2}$ iv) $\frac{1}{2}$

3 i) $\frac{1}{1} = 1$ ii) $\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$ iii) $\frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{2}$

4



graph of sin θ



graph of tan θ

5 i) positive ii) positive iii) negative iv) negative

Ex. 4.B

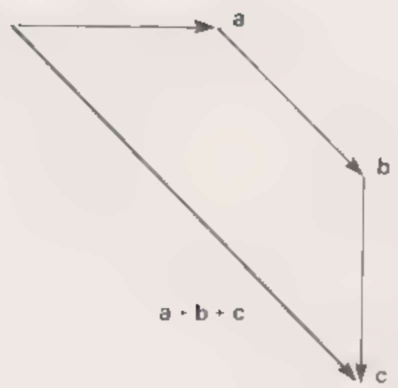
i) 0.492 9 ii) 0.764 4
iii) 0.151 0 iv) 0.955 2
v) 3.000 3 vi) 0.329 1

Ex. 4.C

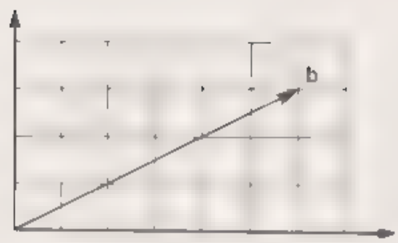
1 $\frac{0.8}{0.6} = \frac{4}{3}$ **2** 0.87 **3** 0.156 4 **4** 0.8

Ex. 4.D

1 i)

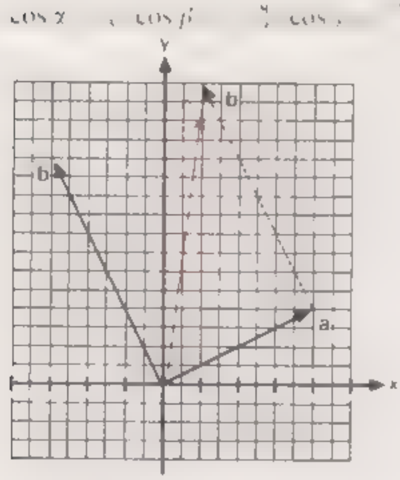


ii)



2 $\cos\alpha$ [illegible] $\cos\beta$ [illegible] $\cos\gamma$ [illegible]

3



Notice how the equivalent **b** has been drawn in order to use the head-to-tail rule.

Section 5 Slope: Area: Growth

A The Slope of a Curve Intro. 5.A

Introduction

In this section we consider how to define the slope of a curve and consider the graphs of the slopes of some very simple graphs.

1 In 3.B.1 we defined the slope of a straight line and in 2.C.5 we defined the tangent of a circle. We now extend these ideas and define the slope of a curve at any point on the curve. 5.A.1

In Fig. 1 $P(a, b)$ is a fixed point on a curve, we aim to define the slope of the curve at P. $Q(x, y)$ is any other point on the curve.

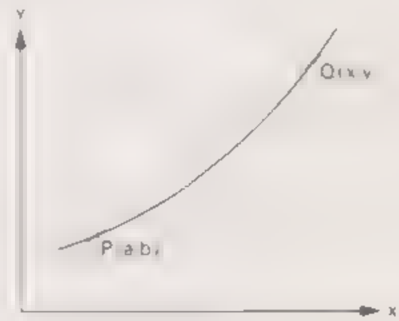


Fig. 1

We assume that the curve does not turn back on itself; i.e. $x \neq a$ for any point Q, except when P and Q coincide.

The straight line joining P and Q is a chord: we have shown PQ extended.

The slope of the straight line PQ is

$$\frac{y-b}{x-a}$$

Now imagine Q rolling down the curve towards P; i.e. x gets closer to a and y gets closer to b. Fig. 2 shows three positions of the extended chord: Q_1P, Q_2P and Q_3P.

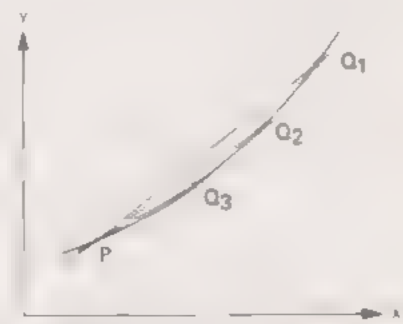


Fig. 2

2 The straight line which *touches* the curve at P is called the tangent to the curve at P. This tangent will have a slope and we will denote it by m. We define the slope of the curve at P to have the same slope as the tangent to the curve at P, i.e. m. The question is: "How do we calculate m?" Clearly we can get an *approximation* to it, by drawing the curve as well as we can, and then drawing in the tangent as well as we can. 5.A.2

3 To see how an algebraic procedure for evaluating m *exactly* can be derived, we note that the tangent at P is the limiting case of the extended chord PQ, as Q tends to P. (Fig. 3.) In symbols 5.A.3

$$\lim_{Q \to P} (\text{chord}) = \text{tangent}$$

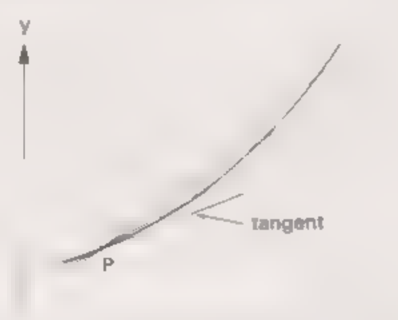


Fig. 3

Similarly for slopes, we can write

$$\lim_{Q \to P} (\text{slope of chord}) = \text{slope of tangent at } P$$
$$= \text{slope of curve at } P$$

or in symbols, more explicitly,

$$\lim_{x \to a} \frac{(y - b)}{(x - a)} = m, \text{ slope of curve at } Q. \qquad (1)$$

4 Since $(y - b)$ is a change in y co-ordinates associated with a change in x co-ordinates $(x - a)$, $(y - b)$ is often denoted by Δy (delta-y) and $(x - a)$ is denoted by Δx (delta-x). A notation used for the slope of a graph of an equation in x and y is $\frac{dy}{dx}$, hence equation (1) can be written as 5.A.4

$$\lim_{\Delta x \to 0} \frac{(\Delta y)}{(\Delta x)} = \frac{dy}{dx}$$

The process of evaluating the limit for particular equations is discussed in G.C.E. A-level mathematics textbooks; the process is called differentiation

5 Although Figs. 1, 2 and 3 show curves with tangents whose slopes are positive, the same reasoning applies to slopes with tangents whose slopes are negative. 5.A.5

6 For an equation such as 5.A.6

$y = x^2 + 2x - 1$

We can evaluate its slope m at each value of x. The set of co-ordinates (x, m) can be plotted and we obtain another graph representing the relationship between x and m, the slope of

$y = x^2 + 2x - 1$

Examples

Here we look at some simple graphs composed of straight lines and plot the graphs of their slopes against x.

1) $y = -\frac{2}{3}x + 2$
has slope $-\frac{2}{3}$.
The slope is represented by the straight line $y = -\frac{2}{3}$ in Fig. 5.

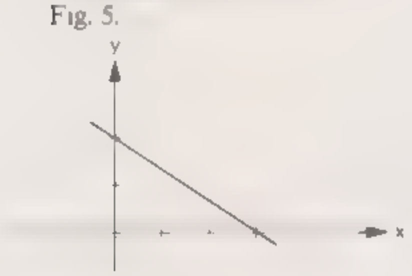


Fig. 4 $y = -\frac{2}{3}x + 2$

Fig. 5 $y = -\frac{2}{3}$

2) The graph of $y = x$ has slope 1 and $y = -x + 2$ has slope -1. Hence the graph of the slope has equation
$y = 1$, x between 0 and 1
$y = -1$, x between 1 and 2

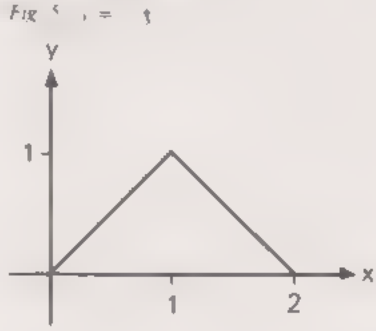


Fig. 6 $y = x$, *x between 0 and 1*
$y = -x + 2$, *x between 1 and 2*

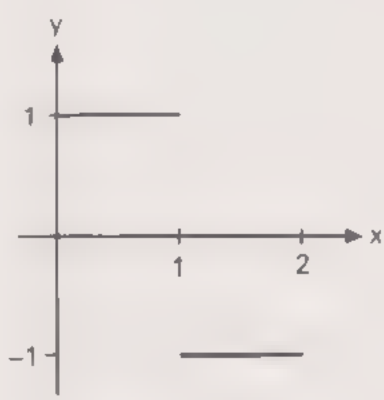
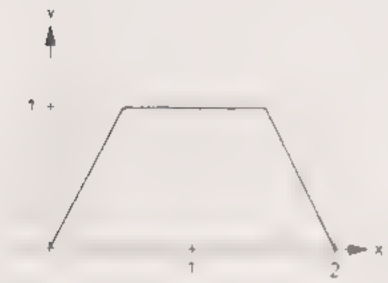
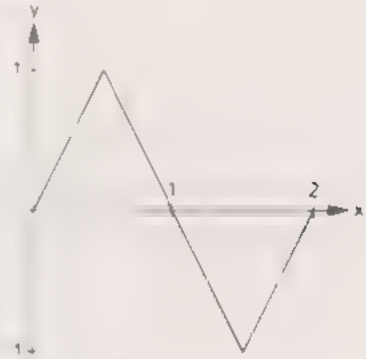


Fig. 7 $y = 1$, *x between 0 and 1*
$y = -1$, *x between 1 and 2*

Exercise Ex. 5.A

Draw graphs of the slopes of the following.





B The Area under a Curve

Introduction Intro. 5.B

In this section we look at a way of approximating the area between a curve and the x-axis, by summing rectangles. We also evaluate the areas between some straight lines and the x-axis and hence relate "slope" and "area".

1 We have seen how to evaluate the area enclosed by triangles and quadrilaterals. How can the area between a curve and the x-axis, for example, be obtained? We shall consider a way of estimating the area between a curve and the x-axis, as shown in Fig. 1. We shall restrict ourselves to curves for which y increases as x increases. 5.B 1

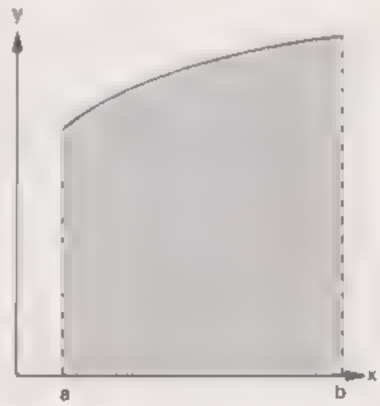


Fig. 1

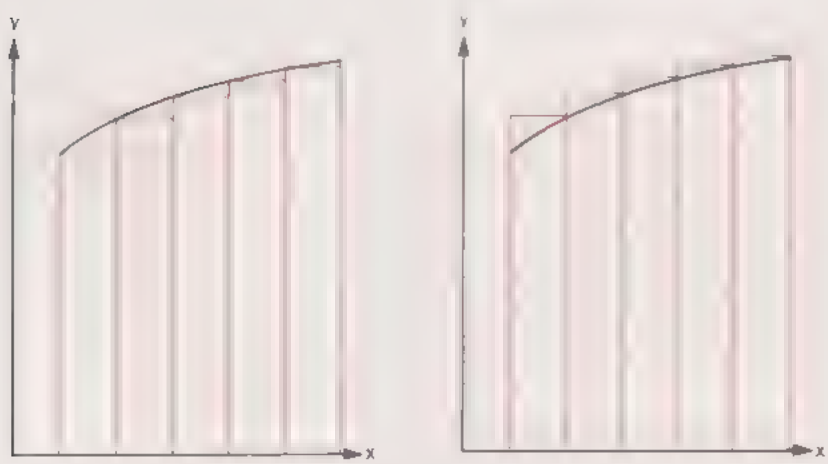


Fig. 2

Fig. 2 shows a curve drawn twice. In each case the interval from a to b on the x-axis is divided into five parts. The sum of the areas of the five red rectangles on the left L_1 is *less than A*, the area between the curve and the x-axis. Whereas the sum of the five red rectangles on the right. U_1, is *greater than A*.

2 Imagine now the interval from a to b divided into twice as many intervals as in Fig. 2. (Fig. 3.) 5.B.2

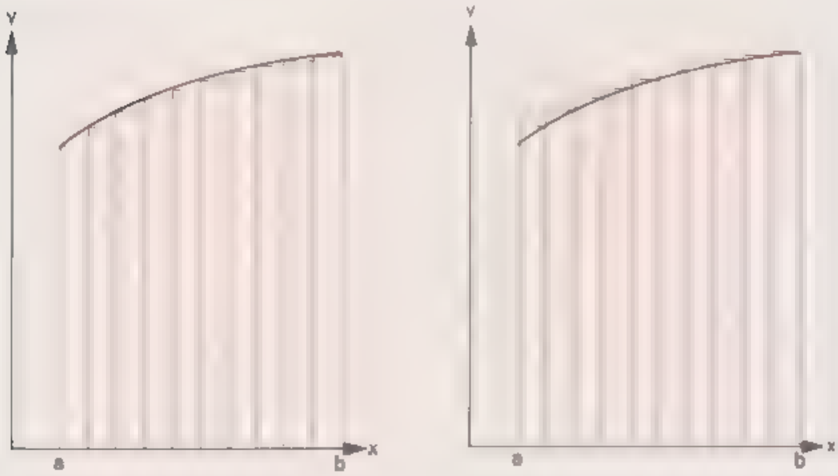


Fig 3

Again the sum of areas of the ten rectangles on the left, L_2 is less than A but it looks a better approximation to A than L_1.

Also the sum of the areas of the ten rectangles on the right, U_2 is greater than A but it looks a better approximation to A than U_2.

Imagine now dividing the interval from a to b into more and more intervals. For each number of intervals we gain two approximations to A by forming rectangles as shown in Fig. 2 and Fig. 3.

If the sequence of the sums of areas of rectangles like those in the left-hand figures is, say

$L_1, L_2, L_3, L_4, \ldots, L_n$

each L will be less than A. Similarly, if the sequence of the sums of the areas of the rectangles like those in the right-hand figures is, say

$U_1, U_2, U_3, U_4, \ldots, U_n$

each U will be greater than A.

In general, the more subdivisions we make; i.e. the larger n is, the closer L_n and U_n are to A. For a particular curve, the L sequence might be

9.81, 9.90, 9.94, 9.97, 9.98, 9.99

and the U sequence might be

10.21, 10.13, 10.09, 10.07, 10.03, 10.02

and we would know that A the area between the curve and the x-axis is sandwiched between 9.99 and 10.02. 5.B.3

3 For a particular curve, we would want to know A *exactly*. The process of integration carries out the process of dividing the interval from a to b into infinitely many subdivisions and hence evaluates A. The mechanics of integration are strikingly simple and it is described in G.C.E. A-level mathematics text books. 5.B.4

4 We next evaluate the areas between two straight lines and the x-axis in order to indicate the relationship between integration (finding areas) and differentiation (finding slopes).

From Fig. 4, we can see that the area between $y = 2$ and the interval from 1 to 3 on the x-axis is 4 units (by counting squares).
Note that, from Fig. 5, the difference between y when $x = 3$ and y when $x = 1$ is also $2 \times 3 - 2 \times 1 = 4$.

Also note that $y = 2$ is the graph of the slope of $y = 2x$.

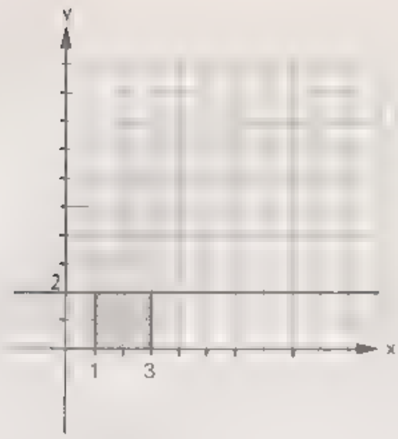


Fig 4 $y = 2$

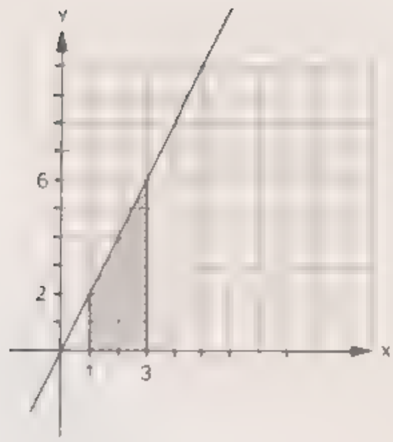


Fig 5 $y = 2x$

From Fig. 5 the area between $y = 2x$ and the interval on the x-axis from 1 to 3 is 8 units (by counting squares and noting that the part squares add to 2 units, since $y = 2x$ cuts pairs of vertical squares diagonally in half).

From Fig. 6, the difference between y when $x = 3$ and y when $x = 1$ is also

$3^2 - 1^2 = 8.$

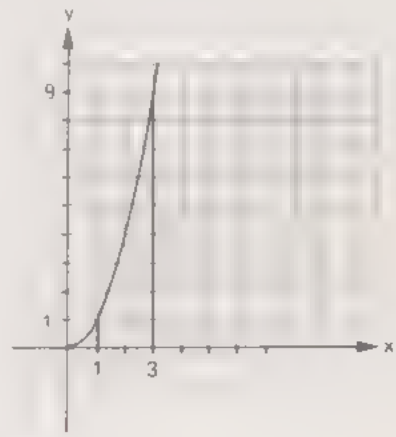


Fig 6 $y = x^2$

By comparing this observation with the fact that $y = 2$ is the graph of the slope of $y = 2x$, we might be tempted to guess that $y = 2x$ is the graph of the slope of $y = x^2$. (In fact, this guess is correct!)

C Exponential Growth and Decay

Introduction

In this section we investigate the equations which represent exponential growth and decay, viz

$y = e^{kt}$

and

$y = e^{-kt}$

Natural logarithms (ln y) and "half-life" are defined.

1 If the relationship between a quantity y and time t can be represented 5.C.1
by an equation involving y and t, the rate of growth (or decay) of y at a point in time is given by the slope of the graph of the equation for that value of t.

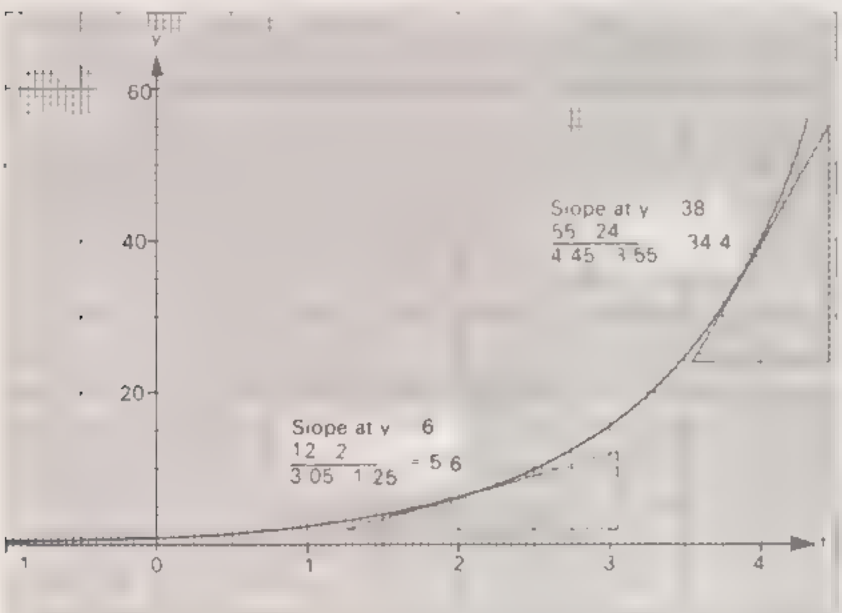


Fig. 1

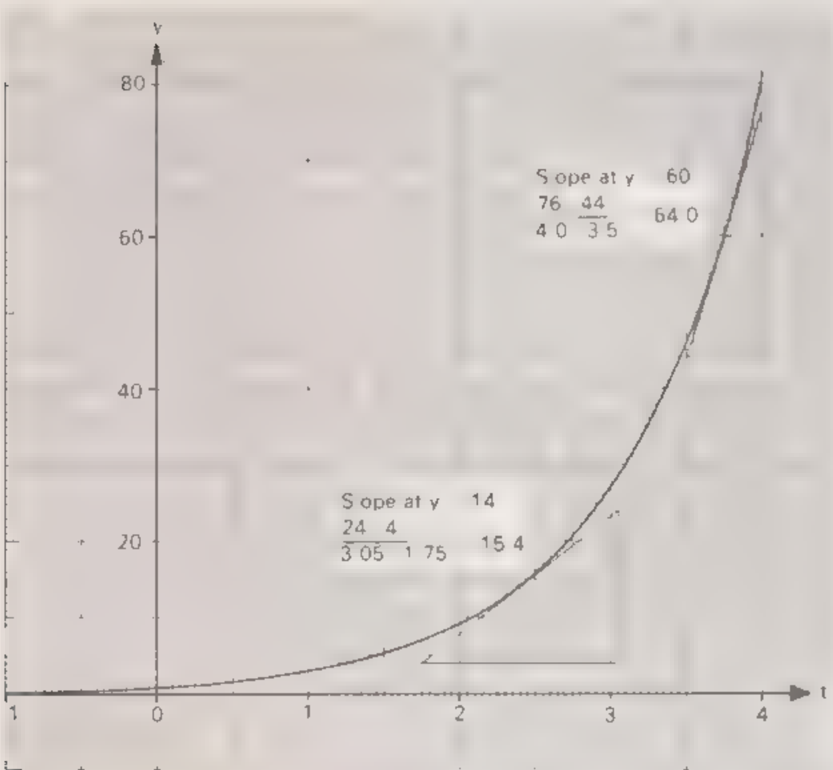


Fig. 2

Fig. 1 and Fig. 2 present the graphs of the equations $y = (2.5)^t$ and $y = (3)^t$ respectively. Note that different scales are used for y and t since in both cases y increases more quickly than t. On Fig. 1 we have estimated the slope (i.e. rate of growth of y) at the points on the curve where $y = 6$ and 38. In fact, for any value of y, the rate of growth is just *less than* the y-value. On Fig. 2 the rate of growth of y has been estimated at $y = 14$ and 60. In both cases, the rate of growth is just greater than the value of y. In fact, for any value of y, the growth rate is just *greater than* the y-value.

This leads us to suspect that there is a number, e, between 2.5 and 3 for which the rate of growth of y for the equation $y = e^t$ is *equal to* the value of y.

2 In fact, e is non-terminating decimal which does not have a recurring pattern in its digits. The value of e to 4 decimal places is 5.C.2

$e = 2.718\,3$

Fig. 3 presents the graph of $y = e^t$

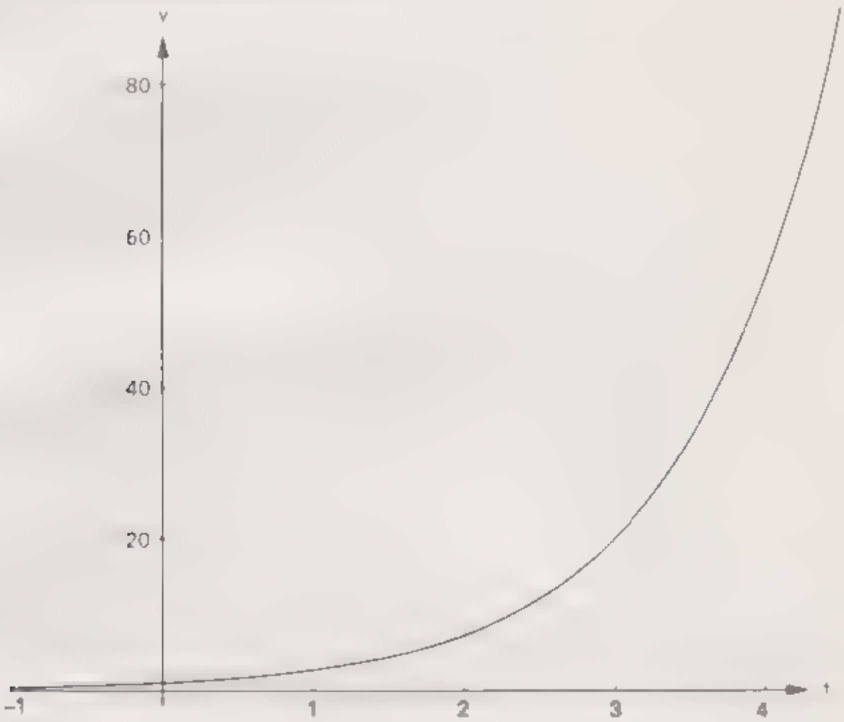


Fig. 3

3 The equation 5.C.3

$y = e^t$

is such that its slope, $\frac{dy}{dt}$, is equal to y for any value of y i.e.

$$\frac{dy}{dt} = y$$

In 1.B.1, we saw that, if $x = a^n$

where a is positive, then

$n = \log_a x$

Similarly, for

$y = e^t$

we have

$t = \log_e y$

i.e. t is the logarithm to base of e of y.

There is a special notation for $\log_e y$; we write

$\log_e y$ as $\ln y$

Logarithms to base e are called natural logarithms; you will find them tabulated on pages 28 and 29 of *Clark's Tables*.

4 If y and t are related by the equation 5.C.4

$$y = e^t$$

y is said to increase exponentially with t.

An equivalent notation for

$$y = e^t$$

is

$$y = \exp(t)$$

Example

Suppose that the rate at which the population of a country is increasing is directly proportional to the size of the population; then, if we denote the population size at any time, t, by y and its rate of growth by $\frac{dy}{dt}$, we may write

$$\frac{dy}{dt} \propto y$$

i.e.

$$\frac{dy}{dt} = ky \qquad (1)$$

where k is a constant of proportionality.

If $k = 1$, (1) becomes

$$\frac{dy}{dt} = y$$

and we may deduce that

$$y = e^t \qquad (2)$$

More generally, it can be shown that, if

$$\frac{dy}{dt} = ky$$

then

$$y = e^{kt} \qquad (3)$$

(See almost any G.C.E. A-level Mathematics Text book.)

Equation (3) has a faster growth than equation (2) if k is greater than 1, but a slower growth if k is less than 1. The following table illustrates this point for $k = 2$, $k = 1$ and $k = \frac{1}{2}$.

t	e^t	e^{2t}	$e^{\frac{1}{2}t}$
0	1	1	1
1	$e \approx 2.718$	$e^2 \approx 7.4$	$e^{\frac{1}{2}} \approx 1.65$
2	$e^2 \approx 7.4$	$e^4 \approx 54.6$	$e \approx 2.718$
4	$e^4 \approx 54.6$	$e^8 \approx 2980$	$e^2 \approx 7.4$

5 Fig. 4 shows the graphs of 5.C.5

$$y = e^{kt}$$

for some k, and (t_1, y_1), (t_2, y_2), (t_3, y_3) are three points on the graph. Since these points lie on the graph,

we have

$$y_1 = e^{kt_1} \qquad (4)$$

$$y_2 = e^{kt_2} \qquad (5)$$

$$y_3 = e^{kt_3} \qquad (6)$$

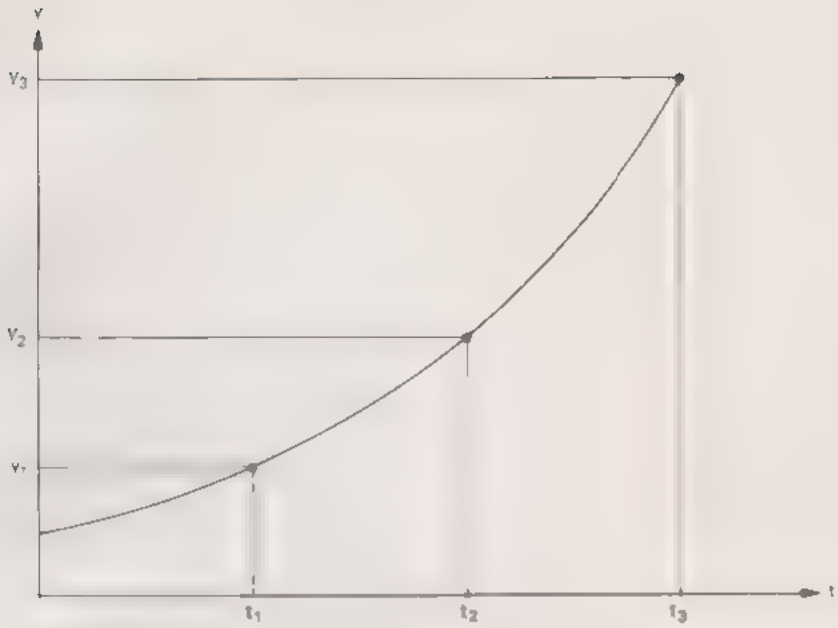


Fig. 4

Divide (5) by (4)

$$\frac{y_2}{y_1} = \frac{e^{kt_2}}{e^{kt_1}} = e^{kt_2} e^{-kt_1}$$

i.e. $$\frac{y_2}{y_1} = e^{k(t_2 - t_1)} \tag{7}$$

Divide (6) by (5)

$$\frac{y_3}{y_2} = \frac{e^{kt_3}}{e^{kt_2}} = e^{kt_3} e^{-kt_2}$$

i.e.

$$\frac{y_3}{y_2} = e^{k(t_3 - t_2)} \tag{8}$$

Now consider the particular case (illustrated in Fig. 4)

$y_2 = 2y_1$

$y_3 = 2y_2$

Hence (7) and (8) can be rewritten as

$$e^{k(t_2 - t_1)} = 2 \tag{9}$$

and

$$e^{k(t_3 - t_2)} = 2 \tag{10}$$

respectively.

The equation

$y = e^{kt}$

can be written as $\ln y = kt$. Hence (9) and (10) become

$k(t_2 - t_1) = \ln 2$

and

$k(t_3 - t_2) = \ln 2$

or

$$(t_2 - t_1) = (t_3 - t_2) = \frac{\ln 2}{k} \tag{11}$$

From (11) we deduce that the t-intervals over which y doubles itself are equal. Such intervals are called characteristic doubling intervals.

Note that the larger is k, the smaller is the characteristic doubling interval.

6 Fig. 4 exhibits exponential growth; exponential decay is just as common in physical problems. A quantity y exhibits exponential decay with time, t, if its rate of growth or slope is negative and is directly proportional to its size: 5.C.6

$$\frac{dy}{dt} \propto -y$$

i.e.

$$\frac{dy}{dt} = -ky \tag{12}$$

where k is positive.

From (12) by analogy with (3), we have

$$y = e^{-kt} \tag{13}$$

Fig. 5 shows the graph of (13) for some k.

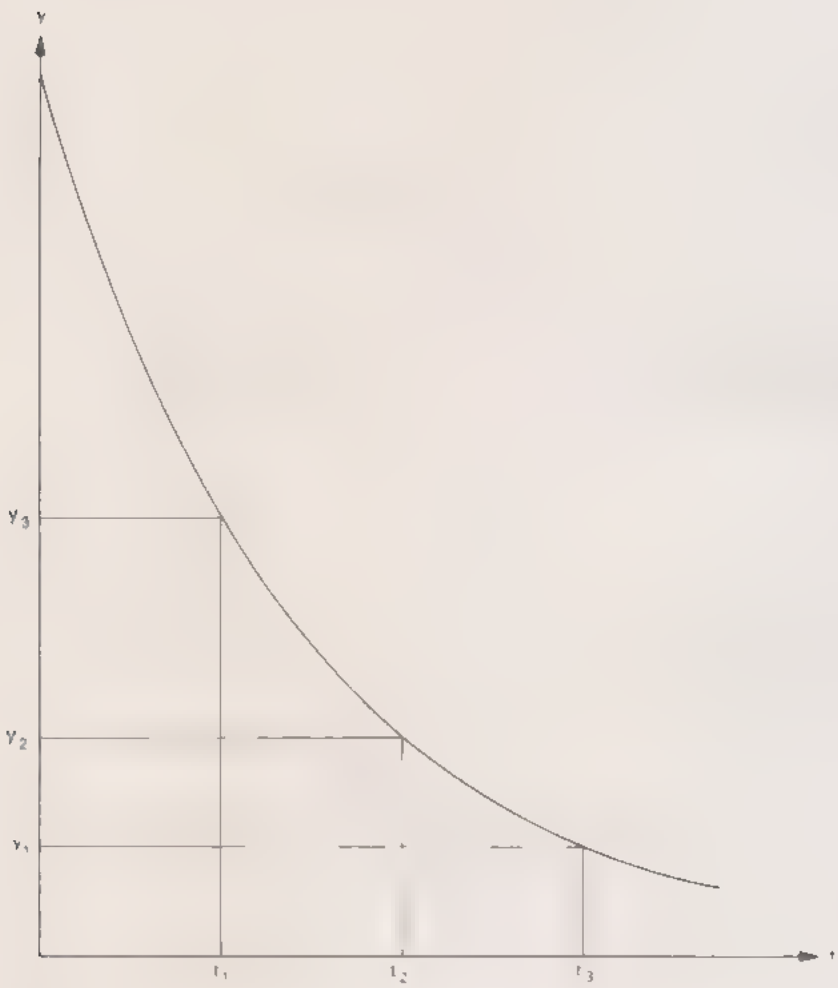


Fig 5

The three points (t_1, y_1), (t_2, y_2) and (t_2, y_3) lie on the graph and we can repeat an argument similar to that in 5.C.5 to obtain a characteristic halving interval.

Since the three points lie on the curve

$$y_1 = e^{-kt_1}$$

$$y_2 = e^{-kt_2}$$

$$y_3 = e^{-kt_3}$$

Hence

$$\frac{y_2}{y_1} = \frac{e^{-kt_2}}{e^{-kt_1}} = e^{-k(t_2-t_1)}$$

and

$$\frac{y_3}{y_1} = \frac{e^{-kt_3}}{e^{-kt_2}} = e^{-k(t_3 - t_2)}$$

In particular if

$$y_2 = \tfrac{1}{2}y_1$$

and

$$y_3 = \tfrac{1}{2}y_2$$

$$e^{-k(t_2 - t_1)} = e^{-k(t_3 - t_2)} = \tfrac{1}{2} = 2^{-1}$$

or

$$e^{k(t_2 - t_1)} = e^{k(t_3 - t_2)} = 2$$

and hence

$$k(t_2 - t_1) = k(t_3 - t_2) = \ln 2$$

i.e.

$$(t_2 - t_1) = (t_3 - t_2) = \frac{\ln 2}{k}$$

These t-intervals in which y halves its value are called characteristic halving intervals. Such a t-interval is often called the "half life" of whatever y represents.

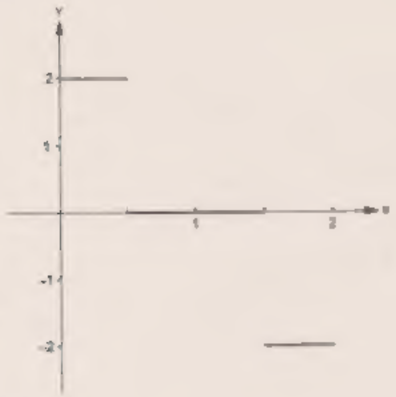
Exercise

There are no exercises for B and C since their function is to whet your appetite, the treatment being brief.

Answers to Exercises for Section 5

Section 5
Answers

Ex. 4.A



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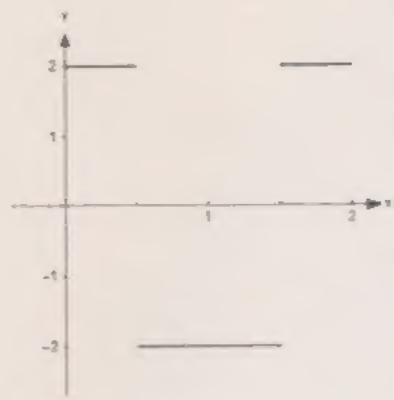

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Acknowledgements

Grateful acknowledgement is made to Oliver and Boyd for permission to reproduce in this Unit extracts from *New Physical and Mathematical Tables* by J. B. Clark, 1969.